Mathematics 552 Homework, March 20, 2020

Here is a summary of part of the main plot, at least as related to analytic functions, to date.

Definition 1. A complex valued function $f(z)$ is ***analytic*** on an open subset D of $\mathbb{C}$ if and only if it is complex differentiable in D. That is for all $z \in D$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists. □

By computing the limit in the definition of $f'(z)$ in two ways, first by letting $\Delta z = \Delta x \to 0$ through real values, and second by letting $\Delta z = i\Delta y \to 0$ go to zero through imaginary values we derived

Theorem 2 (Cauchy-Riemannian equations)**.** *If a function $f(z) = u + iv$ has continuous first partial derivatives in the open set D, then $f(z)$ is analytic if and only if u and v satisfy*

$$u_x = v_y, \qquad u_y = -v_x.$$ □

Then, from our vector calculus class, we recalled

Theorem 3 (Green's Theorem)**.** *Let D be a bounded domain in $\mathbb{C}$ with a nice boundary ∂D. Then if $P(x, y)$ and $Q(x, y)$ are functions on the closure of D that have continuous partial derivatives then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (-P_y + Q_x)\,dx\,dy.$$ □

Green's theorem and the Cauchy-Riemann equations then combine in an easy and natural way to give:

Theorem 4 (Cauchy's Theorem)**.** *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic on the closure of D. Then*

$$\int_{\partial D} f(z)\,dz = 0.$$ □

Remark 5. The proof we gave of the Cauchy's Theorem only used that $f(z) = u + iv$ satisfies the Cauchy-Riemann equations. We will use this to show that conversely if $f(z)$ satisfies the Cauchy-Riemann equations that it is analytic. □

Here are some terms used to describe some subsets of $\mathbb{C}$.

Definition 6. A ***domain*** in $\mathbb{C}$ is a connected open set D. □

Definition 7. A domain is ***simply connected*** if and only if it has no holes in it. (Figure 1 shows some simply connected domains and Figure 2 shows some non-simply connected domains.) □

We used Cauchy's Theorem to show

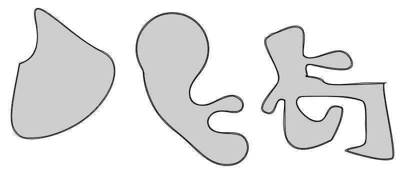

FIGURE 1. Three simply connected domains.

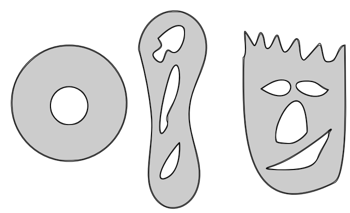

FIGURE 2. Three non-simply connected domains.

Theorem 8. *Let $f(z)$ be analytic in a simply connected domain D. Then $f(z)$ has an antiderivative in D. That is there is a function $F(z)$ defined in D with $F'(z) = f(z)$ in D.* □

Proof. We did this in class when the domain was starlike. The idea in the general case is the same. We will prove the result under the assumption that $f(z) = u + iv$ satisfies the Cauchy-Riemann equations. Choose any $z_0 \in D$ to use as a base point. For any $a \in D$ choose a path from z_0 to a as shown in Figure 3. Then define $F(a)$ by

$$F(a) = \int_\gamma f(z)\,dz.$$

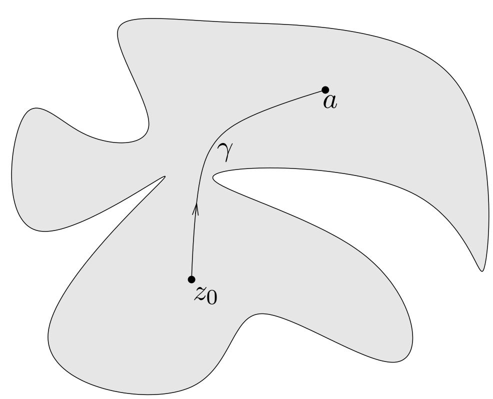

FIGURE 3. Choose a path, γ, from z_0 to a and let $F(a)$ be the line integral $F(a) = \int_\gamma f(z)\,dz$.

(This is basically the same idea where to get the antiderivative of a function you integrate it.) A problem what needs to be worried about is if we use a different path γ_1, do we get the same value for $F(a)$. That is if γ_1 is anther path from z_0 to a is it true that

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma} f(z)\,dz \ ?$$

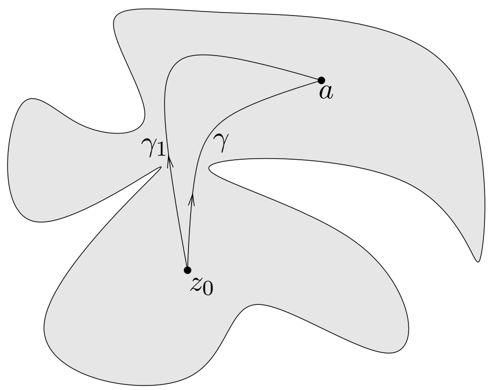

FIGURE 4. Apply the Cauchy Integral formula to the region between γ and γ_1 to conclude $\int_\gamma f(z)\,dz - \int_{\gamma_1} f(z)\,dz = 0$.

Problem 1. Show that if γ and γ_1 are as shown in Figure 4 that

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma} f(z)\,dz$$

holds. *Hint:* Let G be the region between γ and γ_1. Because D is simply connected, and thus has no holes, all of G is contained inside of D and therefore $f(z)$ is analytic on G and ∂G. (This is where we use that D is simply connected.) With our convention that we move along boundaries with the inside on our left, we have that $\partial G = \gamma - \gamma_1$, that is

$$\int_{\gamma} f(z)\,dz - \int_{\gamma_1} f(z)\,dz = \int_{\partial G} f(z)\,dz.$$

Now use use the Cauchy Integral Formula to conclude $\int_{\partial G} f(z)\,dz = 0$. The rest should be easy. $\square$

This comes close to showing that the value of $F(a)$ does not depend on the choice of the choice of γ used in defining $F(a)$. The minor (but very annoying to deal with) gap in the argument is that we used that γ and γ_1 do not intersect other than at their endpoints. We will just ignore this problem and get on with the rest of the proof.

We now need to show that $F'(a) = f(a)$. The definition of the derivative is

$$F'(a) = \lim_{h\to 0} \frac{F(a+h) - F(a)}{h}.$$

To see that this limit exists we first get a formula for $F(a+h) - F(a)$.

Problem 2. Let $[a, a+h]$ be the line segment between a and $a+h$. Let γ and γ_1 be curves as in Figure 5. Use an argument similar to the one used in Problem 1 to show

$$\int_{\gamma} f(z),\, dz + \int_{[a,a+h]} f(z)\, dz - \int_{\gamma_1} f(z)\, dz = 0$$

and therefore

$$F(a+h) - F(a) = \int_{[a,a+h]} f(z)\, dz$$

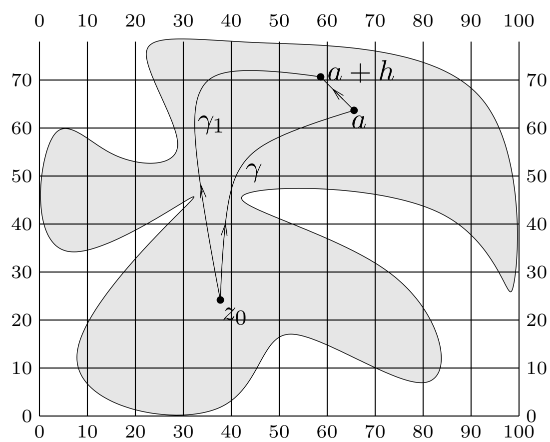

FIGURE 5. To reuse an idea we have just used, apply the Cauchy Integral Theorem to the region bounded by the curves γ, $[a, a+h]$ and γ_1 to conclude $\int_{\gamma} f(z)\, dz + \int_{[a,a+h]} f(z)\, dz - \int_{\gamma_1} f(z)\, dz = 0$. Note this works because the domain is simply connected and therefore the region bounded by these curves is contained in the domain. $\square$

To summarize (and maybe over summarize) what we have done so far toward proving Theorem 8 we have shown that the function $F(a)$ is well defined (which here means that the value) $F(a)$ is independent of the path γ used to define $F(a)$ and that

$$F(a+h) - F(a) = \int_{[a,a+h]} f(z)\, dz.$$

This can be simplified.

Problem 3. Use that $z = a + th$ with $0 \le t \le 1$ parameterizes the segment $[a, a+h]$ to show that

$$F(a+h) - F(a) = h \int_0^1 f(a+th)\, dt$$

and therefore

$$\frac{F(a+th) - F(a)}{h} = \int_0^1 f(a+th)\, dt$$

holds. $\square$

With all this in hand we can not finish the proof of Theorem 8. We wish to find an antiderivative for f. And the guess is that it is F. And in light of the formulas we now have this in not hard to check:

$$\begin{aligned}
F'(a) &= \lim_{h\to 0} \frac{F(a+h)-F(a)}{h} \\
&= \lim_{h\to 0} \int_0^1 f(a+th)\,dt \\
&= \int_0^1 \lim_{h\to 0} f(a+th)\,dt \\
&= \int_0^1 f(a+0)\,dt \\
&= \int_0^1 f(a)\,dt \\
&= f(a)t\Big|_{t=0}^{1} \\
&= f(a).
\end{aligned}$$

Therefore F is an antiderivative of f as required. □

You maybe wondering if it is really necessary that the domain D in Theorem 8 is simply connected. To give examples showing that it is required, we recall the following, which was proven a few weeks ago.

Proposition 9. *Let $f(z)$ be analytic in the domain and let γ be a path in D with beginning point, γ_{begin}, and endpoint, γ_{end}. Assume that $f(z)$ has an antiderivative, $F(z)$, that is $F'(z) = f(z)$ Then*

$$\int_\gamma f(z)\,dz = F(z)\Big|_{\gamma_{\mathsf{begin}}}^{\gamma_{\mathsf{end}}} = F(\gamma_{\mathsf{end}}) - F(\gamma_{\mathsf{begin}}).$$

In particular it γ is a closed curve (that is if γ starts and ends at the same point: $\gamma_{\mathsf{begin}} = \gamma_{\mathsf{end}}$) then

$$\int_\gamma f(z)\,dz = 0$$ □

Or to give a one line synopses:

$$\int_\gamma f(z)\,dz = 0 \qquad \text{(whenever } f \text{ has an antiderivative and } \gamma \text{ is closed.)} \tag{1}$$

Example 10. Let D be the domain $D = \{z \in \mathbb{C} : z \neq 0\}$. Then the function

$$f(z) = \frac{1}{z}$$

has no antiderivative in D. To see this let γ be the circle $|z| = 1$. This is a closed curve. If $f(z)$ has an antiderivative in D, then by Equation (1) we would have

$$\int_{|z|=1} \frac{1}{z}\,dz = 0.$$

But we have computed this integral several times and

$$\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i \neq 0.$$

Thus $f(z)$ does not have any antiderivative in D.

Problem 4. Let D be the ***right half plane***. That is

$$D = \{z = x + iy : x > 0\}.$$

(a) Draw a picture of D and give an explanation of why it is called the "right half plane" and why it is simply connected.

(b) The function $f(z) = \dfrac{1}{z}$ is analytic in D and D is simply connected and therefore by Theorem 8 it has an antiderivative in D. Show that

$$F(z) = \ln(|z|) + i\arctan\left(\frac{y}{x}\right)$$

is an antiderivative of $f(z)$. (Note the function $f(z)$ is defined in D as $x > 0$ on D and this the denominator of y/x is never zero.) $\square$

Theorem 8 has numerous consequences, which we now derive:

Definition 11. Let $f(z)$ be analytic in the domain D. Then $g(z)$ is a ***logarithm*** of $f(z)$ in D if and only if $e^{g(z)} = f(z)$. $\square$

Problem 5. Explain why this is the proper definition of $g(z)$ being a logarithm of $f(z)$. $\square$

Problem 6. Show that if $g(z)$ is a logarithm of $f(z)$ in D then for any integer n the function $h(z) = g(z) + 2\pi ni$ is also a logarithm of $f(z)$. Thus logarithms are never unique. $\square$

There is an easy condition that implies that $f(z)$ has a logarithm.

Theorem 12. *Let D be a simply connected domain and $f(z)$ a function that is analytic in D and nonvanishing in D (that is $f(z) \neq 0$ for all $z \in D$). Then $f(z)$ has a logarithm in D.*

Problem 7. Prove this along the following lines. (This proof is motivated by noting that if $g(z) = \log f(z)$ then we should have $g'(z) = f'(z)/f(z)$.)

(a) We have shown that if $f(z)$ is analytic, then so is the derivative $f(z)$. Therefore $\dfrac{f'(z)}{f(z)}$ is analytic in D. *Hint:* You only have to copy this down fact down.

(b) Explain why $\dfrac{f'(z)}{f(z)}$ has an anti-derivative. Call this anti-derivative $g_1(z)$. *Hint:* Theorem 8.

(c) Show that $f(z)e^{-g_1(z)}$ is constant. *Hint:* About the most natural way to show that a function is constant is to show its derivative is zero. Note that
$$\frac{d}{dz}\left(f(z)e^{-g_1(z)}\right) = f'(z)e^{-g_1(z)} - f(z)g_1'(z)e^{-g_1(z)}.$$
and, as $g_1(z)$ is an anti-derivative of $f'(z)/f(z)$
$$g_1'(z) = \frac{f'(z)}{f(z)}.$$

(d) From part (c) we have $f(z)e^{-g_1(z)} = c$ for some non-zero complex constant c. Thus $f(z) = ce^{g_1(z)}$. Show that there is a complex constant a so that $g(z) = g_1(z) + a$ is a logarithm of $f(z)$. □

We can also take roots in simply connected domains.

Theorem 13. *Let $f(z)$ be analytic and nonvanishing in the simply connected domain D and let n be a positive integer. Then there is an analytic function $h(z)$ with*
$$h(z)^n = f(z).$$
(Thus when $n = 2$, $h(z)$ is a square root of $f(z)$, when $n = 3$, $h(z)$ is a cube root of $f(z)$ etc.)

Problem 8. Prove this. *Hint:* Let $g(z)$ be a logarithm of $f(z)$ and consider the function $h(z) = e^{g(z)/n}$. □