Mathematics 552 Homework, April 1, 2020

Here is a review of some of part of what we have done in the last couple of classes. Let $f(z)$ be analytic in the disk $|z - z_0| = R$. Then by the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=R} \frac{f(w)}{w-z}\, dw \tag{1}$$

holds when z is in the disk $B(z_0, R) = \{z : |z - z_0| < R\}$. We then took the fraction $1/(w - z)$ and expanded in a as a geometric series in powers of $(z - z_0)$ as follows (with some of steps we did in class skipped here)

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)} \left(\frac{1}{1 - \dfrac{z-z_0}{w-z_0}} \right) \\
&= \frac{1}{(w-z_0)} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}.
\end{aligned}$$

This converges when the ratio is has size < 1, that is if

$$\left| \frac{z-z_0}{w-z_0} \right| < 1.$$

This is equivalent to $|z - z_0| < |w - z_0|$. But in the Cauchy Integral Formula (1) we have that w is on the circle $|w - z_0| = R$ and thus the series converges for $|z - z_0| < R$, that is for all $z \in B(z_0, R)$. Using the convergent series in the Cauchy Integral Formula find that for $|z - z_0| < R$

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|w-z_0|=R} \frac{1}{w-z} f(w)\, dw \\
&= \frac{1}{2\pi i} \int_{|w-w_0|=R|} \left(\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} \right) f(w)\, dw \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|w-z_0|=R} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \right) (z-z_0)^n \quad \text{(interchange sum and integration)} \\
&= \sum_{n=0}^{\infty} c_n (z-z_0)^n,
\end{aligned}$$

where

$$c_n = \frac{1}{2\pi i} \int_{|w-z_0|=R} \frac{f(w)\, dw}{(w-z_0)^{n+1}}.$$

Recalling that we have a variant on the Cauchy Integral Formula for the derivatives of f which in this case gives

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|w-z_0|=R} \frac{f(w)\,dw}{(w-z_0)^{n+1}}$$

which gives us anther formula for the coefficient c_n

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

This is the form of the coefficient that you mostly likely saw in your Math 142 (or equivalent) class.

One of the main takeaways from this is that if $f(z)$ is analytic in a disk $B(z_0, R)$, then it has a convergent power series expansion

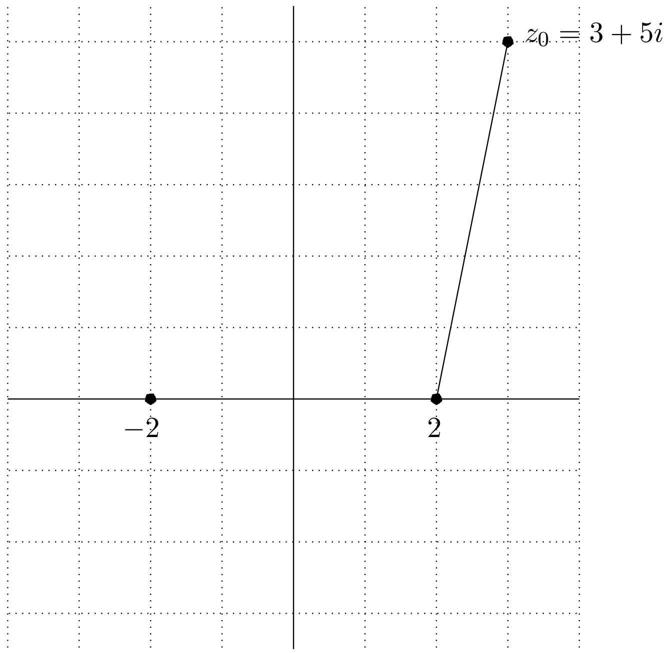
$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{2}$$

in this disk. If we define a ***singularity*** of the function $f(z)$ to be a point b where either $f(z)$ is undefined or is not analytic then what we have just shown is that if $f(z)$ is analytic at z_0 and R is the distance of z_0 to the singularity of f nearest to z_0, then R is the radius of convergence of the (2). In many cases this makes it possible to to find the radius of convergence of a series without doing any calculations with the ratio, root, or other convergence tests.

For example consider the function

$$f(z) = \frac{e^{z^2}}{z^2 - 4}.$$

This a analytic at all points other that $z = 2$ and $z = -2$ where the denominator becomes zero. So if $f(z)$ is expanded around $z_0 = 0$, that is as $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then the singularities -2 and 2 are both at a distance of $R = 2$ form z_0 and therefore the radius of convergence is $R = 2$ as this is the smallest distance of z_0 to a singularity. If we wish to expand $f(z)$ around the point $z_0 = 3 + 5i$ then, looking at the following figure,



we see that the singularity of f closest to z_0 is 2 and that the distance of z_0 to 2 is $R = \sqrt{1^2 + 5^2} = \sqrt{26}$. Therefore the $R = \sqrt{26}$ is the radius of convergence.

Problem 1. For the function

$$h(z)\frac{z^2 - 1}{z^4 - 81}$$

(a) Find the singularities of $h(z)$.
(b) Find the radius of convergence when $h(z)$ is expanded around $z_0 = 0$.
(c) Find the radius of convergence when $h(z)$ is expanded around $z = 3 + 5i$. □

We now look at anther consequence of the Cauchy Integral Formula. Befine $B(z, r)$ to be the open disk of radius r centered at z. Explicitly

$$B(z, r) = \{w : |w - z| = r\}.$$

Then $\partial B(z, r)$ is the circle in the w plane defined by $|w - z| = r$.

Definition 1. Let f be the continuous on the circle $|w - z| = r$. Then the ***average value*** of f on this circle is

$$\text{Average of } f \text{ on } \partial B(z, r) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt.$$

Problem 2. Show that if $f(z) = c$ is a constant then for any disk

$$\text{Average of } f \text{ on } \partial B(z, r) = c.$$

□

Problem 3. Let f be analytic in the closure of the disk of radius r about z. Then from the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{|w-z|=r} \frac{f(w)\,dw}{w-z}.$$

Use the parametrization $w = z + re^{it}$ with $0 \le t \le 2\pi$ to show

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt$$

and thus the value $f(z)$ is the average of f on the circle of radius r about z. This is the ***mean value property*** of analytic functions. □

Proposition 2. *Let $f = u + iv$ be analytic in a connected domain D. Assume that $|f(z)|$ is constant. Then $f(z)$ is constant.*

Problem 4. Prove this along the following lines.

(a) If $|f(z)|$ is constant then show $u^2 + v^2 = c$ for some real constant c.
(b) If $c = 0$ show $f(z)$ is the constant function 0.
(c) If $c \ne 0$ use the Cauchy-Riemann equations to show $f(z)$ is constant. To be a bit more explicit if $u^2 + v^2 = c$, that taking the partial derivative $\frac{\partial}{\partial x}$ gives

$$\frac{\partial}{\partial x}(u^2+v^2) = 2uu_x + 2vv_x = 0.$$

and likewise

$$\frac{\partial}{\partial y}(u^2+v^2) = 2uu_y + 2vv_y = 0.$$

so we end up with system

$$\begin{aligned} uu_x + vv_x &= 0 \\ uu_y + vv_y &= 0. \end{aligned}$$

Now use the Cauchy-Riemann Equations in these equations to show that $u_x = u_y = v_x = v_y = 0$. This implies that u and v are constant and therefore that $f = u + iv$ is constant. □

The following is a variant of the triangle inequality for integrals.

Proposition 3. *Let $f(z)$ be continuous on the circle $|z - z_0| = r$. Then*

$$\left|\int_0^{2\pi} f(z_0+re^{it})\,dt\right| \le \int_0^{2\pi} |f(z_0+re^{it})|\,dt$$

and if equality holds, then $|f(z_0+re^{it})|$ is constant (as a function of t.) □

Theorem 4 (Maximum modulus principle)**.** *Let $f(z)$ be analytic on the closure of $B(z_0, R)$ and assume that $|f(z)|$ has a maximum at $z = z_0$ (that is $|f(z)| \le |f(z_0)|$ for $z \in B(z_0, R)$). Then $f(z)$ is constant in $B(z_0, R)$.*

Problem 5. Prove this along the following lines.

(a) If $0 < r < R$ use the mean value property of analytic functions to write
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt.$$
(You don't have to prove again as you did it above). Now use Proposition 3
$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\,dt \le |f(z_0)|$$
and that this implies $|f(z_0 + re^{it})| = |f(z_0)|$ for $0 \le t \le 2\pi$.

(b) By varying $r \in (0, R)$ and $t \in [0, 2\pi]$ in part (b) show that $|f(z)| = |f(z_0)|$ for $z \in B(z_0, R)$.

(c) Now use Proposition 2 to show $f(z)$ is constant in $B(z_0, R)$.

Here is anther form of the maximum modulus principle.

Problem 6. Let D be a bounded domain and let $f(z)$ be analytic on $\overline{D}$ (the closure of D.) Then $f(z)$ achieves its maximum on the boundary, ∂D, of D.

Problem 7. Prove this along the following lines.

(a) If $|f(z)|$ is constant, then $f(z)$ is constant and so the maximum of $|f(z)|$ occurs at all points of ∂D. In particular it occurs on the boundary.

(b) So assume that $f(z)$ is not constant. Assume, toward a contradiction that the maximum of $|f(z_0)|$ occurs in D rather than on ∂D. Then get a contradiction by showing that $f(z)$ is constant.

Proposition 5 (Minimum modulus principle)**.** *Let $f(z)$ be analytic on the closure of $B(z_0, R)$ and assume that $|f(z)|$ has a minimum at $z = z_0$ (that is $|f(z)| \ge |f(z_0)|$ for $z \in B(z_0, R)$). Then either $f(z)$ is constant or $f(z_0) = 0$.*

Problem 8. Prove this. *Hint:* If $f(z_0) \ne 0$ then show $f(z) \ne 0$ for all $z \in D(z_0, R)$ and then apply the maximum modulus principle to $g(z) = 1/f(z)$.