Mathematics 552 Homework, April 11, 2020

Fist some remarks. LaTeX has a large number of mathematical functions built in. So there is no problem of typesetting

$$\sin\theta, \quad \ln(t), \quad \min(a,b), \quad \cosh(x)$$

as

```
$$
\sin \theta, \quad \ln(t),\quad \min(a,b), \quad \cosh(x)
$$
```

(The `\quad` is a unit of spacing, used here to put some white space between the formulas.) Note that `sin t` and `\sin t` are note the same. We check this by seeing that

```
$$
sin t \quad \sin t
$$
```

gives

$$sin t \quad \sin t$$

and we see that `\sin t` is the correct way to typeset $\sin t$. You can find a list of LaTeX's included functions at `https://www.overleaf.com/learn/latex/Operators`. But you might want to use a function that is not in the standard list. In particular in this homework we will be working with residues and want to be able to typeset expressions such as $\operatorname{Res}(f,a)$. So we have to define a new function, which is usually done in the preamble of the document. Here is an example:

```
\documentclass[12pt]{amsart} %% Use the AMS article class with 12 point type.
%  The preamble is the stuff before \begin{document}

% Here is how to define our function \Res
\newcommand{\Res}{\operatorname{Res}}

\begin{document}

The residue of $\tan(z)$ at $z= \pi/2$ is
$$
\Res(\tan z , \pi/2) = -1.
$$

\end{document}
```

You can find more about using `\newcommand` to define new functions at `https://www.overleaf.com/learn/latex/Commands`.

Before going to actual mathematics one other LaTeX feature, being able to do multi-line formulas such as

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= aa+ab+ba+aa\\ &= a^2+2ab+b^2.\end{aligned}$$

You can find several methods for doing this at
`https://www.overleaf.com/learn/latex/Aligning%20equations%20with%20amsmath`.
The code I used for the about was

```
\begin{align*}
(a+b)^2&= (a+b)(a+b)\\
       &= aa +ab + ba + aa\\
       &= a^2+2ab+b^2.
\end{align*}
```

For this to work you have to be using the `msrt`* document class, rather than the basic `article` class.

Homework due Monday at 5:00pm
Type up the following in LaTeX

Name: *Your name.*

Here we look at some applicators of the residue theorem. Recall that if we have a fraction

$$f(z) = \frac{g(z)}{h(z)}$$

then at a point $z = a$ where $h(a) = 0$ and $h'(a) \neq 0$ then $f(z)$ has a simple pole at $z = a$ and the residue of $f(z)$ at $z = a$ is

$$\operatorname{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

We use this to evaluate integrals of the form

$$\int_0^{2\pi} R(\cos t, \sin t)\, dt$$

where $R(x, y)$ is a rational function of x and y. The idea is to let

$$z = e^{it}$$

with $0 \le t \le 2\pi$. Then

$$\begin{aligned}\cos t &= \frac{e^{it} + e^{-it}}{2} \\ &= \frac{z + z^{-1}}{2} \\ &= \frac{z^2 + 1}{2z} \\ \sin t &= \frac{e^{-it} - e^{-it}}{2i} \\ &= \frac{z - z^{-1}}{2i} \\ &= \frac{z^2 - 1}{2iz}.\end{aligned}$$

Also

$$dz = ie^{it}\,dt = iz\,dt$$

and therefore

$$dt = \frac{dz}{iz}.$$

Also as t goes from 0 to 2π the variable $z = e^{it}$ moves over the unit circle defined by $|z| = 1$. Therefore with this change of variable we have

$$\int_0^{2\pi} R(\cos t, \sin t)\,dt = \int_{|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}.$$

Here is an example. Let $a > 1$ and let us compute

$$I(a) = \int_0^{2\pi} \frac{\cos t}{a + \cos t} dt.$$

Using the substitution $z = e^{it}$ this becomes

$$\begin{aligned}I(a) &= \int_{|z|=1} \frac{\frac{z^1+1}{2z}}{a + \frac{z^2+1}{2z}} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{z^2+1}{2z} \frac{1}{a + \frac{z^2+1}{2z}} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{z^2+1}{(2az + z^2 + 1)} \frac{dz}{iz} \\ &= \frac{1}{i} \int_{|z|=1} \frac{z^2+1}{z(z^2+2az+1)} dz\end{aligned}$$

The singularities of the integrand

$$f(z) = \frac{z^2+1}{z(z^2+2az+1)}$$

are where the denominator is zero. That is when $z = 0$ or $z^2 + 2az + 1 = 0$. The easiest way to solve the later is by completing the square. The equation is equivalent to

$$z^2 + 2az + 1 = (z+a)^2 - a^2 + 1 = 0$$

and so

$$(z+a)^2 = \sqrt{a^2-1}$$

and therefore

$$a = -a \pm \sqrt{a^2-1}.$$

Of these two roots one is $-a - \sqrt{a^2-1} < -1$ and therefore is not in the circle $|z| = 1$. The other root is

$$\beta = -a + \sqrt{a^2-1}$$

which is inside of the unit circle. We now compute the residues. Our function is

$$f(z) = \frac{z^2+1}{z(z^2+2az+1)} = \frac{g(z)}{h(z)}.$$

with $g(z) = z^2+1$ and $h(z) = z(z^2+2az+1) = z^3+2az^2+z$. We will also need the derivative of $h(z)$ which is

$$h'(z) = 3z^2 + 4az + 1.$$

So the residue at $z = 0$ is

$$\operatorname{Res}(f, 0) = \frac{g(0)}{h'(0)} = \frac{1}{1} = 1.$$

The residue at $z = \beta$ is

$$\operatorname{Res}(f, \beta) = \frac{g(\beta)}{h'(\beta)} = -2a\sqrt{a^2-1}.$$

(A lot of algebra was skipped in simplifying $g(\beta)/h'(\beta)$.) And now we are pretty much done:

$$\begin{aligned} I(a) &= \frac{1}{i}\int_{|z|=1} \frac{z^2+1}{z(z^2+2az+1)}dz \\ &= \frac{1}{i}2\pi i\Big(\operatorname{Res}(f,0) + \operatorname{Res}(f,\beta)\Big) \\ &= 2\pi\big(1 - 2a\sqrt{a^2-1}\big) \end{aligned}$$