First collection of Final Problems in Math 552.

Send me the solutions to these problems by 5:00pm on Friday April 17. Use as the subject line

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Subject:  Problems 1 to 3, <your name>.
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Your solution should be LaTeX output. I will be happy to answer questions related to these question on Wednesday and Friday. And if you have LaTeX questions you can e-mail be about them (use the word "LaTeX" in the subject line if you are doing this)

Recall that an ***entire function*** is a function $f\colon \mathbb{C} \to \mathbb{C}$ which is analytic at all points of $\mathbb{C}$. One of the more famous results in complex analysis is

Theorem 1 (Liousville's Theorem)**.** *A bounded entire function is constant. (To be explicit if $f(z)$ is analytic and that is a constant M so that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then $f(z)$ is constant.)* □

We can use this to prove what looks like a stronger statement. First a definition (and recall that $B(a,r) = \{z : |z-a| < r\}$ is the open disk of radius r around a).

Definition 2. A subset $S \subseteq \mathbb{C}$ is ***dense*** in $\mathbb{C}$ if and only if for all $a \in \mathbb{C}$ and $r > 0$ the intersection $S\colon B(a,r) \neq \varnothing$. (Put differently, S is dense if and only if every open disk $D(a,r)$ contains at least one point of S.) □

Thus a dense set is large in the sense that it gets close to every point of $\mathbb{C}$.

Theorem 3. *If $f(z)$ is a non-constant entire function, then its range*

$$f[\mathbb{C}] = \{f(z) : z \in \mathbb{C}\}$$

is dense in $\mathbb{C}$.

Problem 1. Prove this. *Hint:* Towards a contradiction assume that this is false. Then there is a non-constant entire function $f(z)$ and a disk $B(a,r)$ such that the disk does not contain any point $f(z)$. Now argue

(a) This implies $|f(z) - a| \geq r$ for all $z \in \mathbb{C}$.

(b) Let

$$g(z) = \frac{1}{f(z) - a}$$

and explain why $g(z)$ is an entire function.

(c) Show that $g(z)$ is bounded.

(d) Finally use Liousville's Theorem to show that $g(z)$ is constant and explain why this contradicts our assumption that $f(z)$ is not constant.

□

We have recently proven:

Theorem 4. *If $f(z)$ has an isolated singularity at $z = a$ and there is a $C > 0$ such that for some $r > 0$*

$$|f(z)| \leq D \qquad \text{for all } z \text{ with } 0 < |z - a| < r,$$

then the singularity is removable. (That is if a function is bounded near an isolated singularity, then the singularity is removable.) □

We now use this to prove

Theorem 5. *Let $f(z)$ have an isolated singularity at $z = a$ and assume*

$$\lim_{z \to a} |f(z)| = \infty.$$

Then $f(z)$ has a pole at a. That is there is an analytic function $f_0(z)$ defined near a and a positive integer $m \geq 1$ such that

$$f(z) = \frac{f_0(z)}{(z - a)^m}$$

and $f_0(a) \neq 0$.

Problem 2. Prove this along the following lines. Let

$$g(z) = \frac{1}{f(z)}$$

(a) Explain why

$$\lim_{z \to a} g(z) = 0.$$

(b) Explain why $g(z)$ has an isolated singularity at $z = a$.
(c) Show that $g(z)$ is bounded near $z = a$ and then explain why this implies that the singularity of $g(z)$ is removable.
(d) Then $z = a$ is a zero of $g(z)$, say it is a zero of order $m \geq 1$, so that

$$g(z) = (z - a)^m g_0(z)$$

where $g_0(z)$ is an analytic function with $g_0(a) \neq 0$ (you do not have to prove this about g). Show that this implies

$$f(z) = \frac{f_0(z)}{(z - a)^m}$$

where $f_0(z) = \dfrac{1}{g_0(z)}$ and why $f_0(z)$ satisfies the conclusion we require.

□

Essential singularities have a more spectacular property.

Theorem 6. *Let $z = a$ be an essential singularity of $f(z)$. Then for any $\delta > 0$ the set*

$$f[D(a, \delta)] = \{f(z) : 0 < |z - a| < \delta\}$$

is dense in $\mathbb{C}$.

Problem 3. Prove this. *Hint:* The most natural proof of this is very much like the proof of Theorem 3: Start by assuming, towards a contradiction, that the result is false, then there is disk $B(b, r)$ such that $f[B(a, \delta)]$ does not intersect $B(b, r)$. Use this to show that

$$g(z) = \frac{1}{f(z) - b}$$

has a removable singularity at $z = a$ and this implies that the singularity of $f(z)$ at $z = a$ is either removable, or a pole, contradicting that $z = a$ is an essential singularity. □