Mathematics 552 Homework, April 18, 2020

In this homework set we are going to look at ***Möbius Transformations*** (also called ***linear fractional transformations*** which are maps $T\colon \mathbb{C} \to \mathbb{C}$ of the form

$$T(z) = \frac{az+b}{cz+d} \quad \text{where } ad - bc \neq 0.$$

The geometry of these maps is quite interesting. Here is a short vido (less than 3 minutes) that shows a bit of the geometry and has some nice graphics. What we will be looking at is how Möbius transformations interact with the geometry of circles.

First we need to fix one problems with Möbius transformations, and that is that the denominator can be zero. We do this by adding an extra number, ∞, to the complex plane. We assume that it has the following properties:

$$\begin{aligned}
\frac{a}{0} &= \infty && \text{when } a \neq 0. \\
\frac{a}{\infty} &= 0 && \text{for } a \neq 0. \\
\infty \cdot a &= \infty && \text{for } a \neq 0 \\
\infty + a &= \infty && \text{for all } a \in \mathbb{C} \\
\infty \cdot \infty &= \infty
\end{aligned}$$

and

$$\begin{aligned}
0 \cdot \infty &= \text{undefined} \\
\infty \pm \infty &= \text{undefined.}
\end{aligned}$$

The reason we do this is so that our Möbius transformation

$$T(z) = \frac{az+b}{cz+d}$$

is defined at all points. For example if $c \neq 0$ we have

$$T(-d/c) = \frac{a(-d/c)+b}{c(-d/c)+d} = \frac{-ad/c+b}{0} = \infty.$$

And

$$T(\infty) = \frac{a(\infty)+b}{c(\infty)+d} = \frac{a+b/\infty}{c+d/\infty} = \frac{a+0}{c+0} = \frac{a}{c}.$$

Maybe a better way to think of this, at least formally, is

$$T(\infty) = \lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \lim_{z\to\infty} \frac{a+b/z}{c+d/z} = \frac{a+0}{c+0} = \frac{a}{c}.$$

We given a name to $\mathbb{C}$ with ∞ added,

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

and call it the ***expended complex plane*** or the ***Riemann sphere.***

Our first result about Möbius transformations is that the composition of two of them is still Möbius transformation.

Theorem 1. *Let $S(z)$ and $T(z)$ be the Möbius transformations*

$$S(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \qquad T(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

then

$$S \circ T(z) = S(T(z)) = \frac{az + b}{cz + d}$$

where

$$\begin{aligned} a &= a_1 a_2 + b_1 c_2 \\ b &= a_1 b_2 + b_1 d_2 \\ c &= c_1 a_2 + d_1 c_2 \\ d &= c_1 b_2 + d_1 d_2 \end{aligned}$$

Problem 1. Prove this. □

Remark 2. If you know about matrix multiplication, it is worth noting that the numbers a, b, c, d are given by matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

□

Theorem 3. *If $S(z)$ and $T(z)$ are the Möbius transformations*

$$S(z) = \frac{az + b}{cz + d}, \qquad T(z) = \frac{dz - b}{-cz + a}$$

then S and T are inverse to each other:

$$S(T(z)) = z, \qquad T(S(z)) = z$$

for all z.

Problem 2. Prove this. □

We now look at how to recognize the equations of circles. The definition of a circle with center $a \in \mathbb{C}$ and with radius is the set of points that have distance r from a. That is it is the set of all points z such that

$$|z - a| = r.$$

Some equations that to not initially look like can be rewritten in this form.

Proposition 4. *Let $a, b \in \mathbb{C}$ with $a \neq b$ and let $\rho > 0$ with $\rho \neq 1$. Then the equation*

$$|z - a| = \rho |z - b|$$

can be rewritten as

$$\left| z - \frac{a - \rho^2 b}{1 - \rho^2} \right| = \frac{\rho |a - b|}{|1 - \rho^2|}$$

and therefore is a circle with center $\dfrac{a - \rho^2 b}{1 - \rho^2}$ and radius $r = \dfrac{\rho |a - b|}{|1 - \rho^2|}$

Proof. A (somewhat complicated) exercise in completing the square. For fixed points a and b the collection of circles gotten by varying ρ are the ***Circles of Apollonius*** which are famous enough to have a Wikipedia article. $\square$

The last Proposition leaves out $\rho = 1$, but that is the easy case.

Proposition 5. *The equation*

$$|z - a| = |z - b|$$

can be rewritten as

$$2\,\mathrm{Re}\left((\bar{b} - \bar{a})z\right) = |b|^2 - |a|^2$$

and this is the equation of a straight line.

Problem 3. Prove this. *Hint:* I found that the easiest way was to first show

$$|z - a|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{a}z) + |a|^2,$$
$$|z - b|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{b}z) + |b|^2.$$

$\square$

To summarize the main points just verified:

Theorem 6. *If $a, b \in \mathbb{C}$ with $a \neq b$ and $\rho > 0$ then the set of points $z \in \mathbb{C}$ with*

$$|z - a| = \rho|z - b|$$

is either a circle ($\rho \neq 1$ or a line ($\rho = 1$). $\square$

This can be used to show that more complicated looking equations define circles or lines. For example let us consider the equation

$$\left|\frac{2z - 3i}{4z + 5} - 6\right| = 7.$$

We first put the fraction over a common demoninator

$$\left|\frac{2z - 3i - 6(4z + 5)}{4z + 5}\right| = \left|\frac{-22z - (30 + 3i)}{4z + 5}\right| = 7.$$

Multiply by $|4z + 5|$

$$| - 22z - (30 + 3i)| = 7|4z + 5|$$

and factor out the coefficients of the z terms

$$22\left|z - \frac{-30 - 3i}{22}\right| = 7 \cdot 4\left|z - \frac{-5}{4}\right|$$

and finally divide by 22

$$\left|z - \frac{-30 - 3i}{22}\right| = \frac{14}{11}\left|z - \frac{-5}{4}\right|$$

which is of the form $|z - a| = \rho|z - b|$ so we see the equation defines a circle (as $\rho = 14/11 \neq 1$).

This example generalizes:

Proposition 7. *Let T be the Möbius transformation*

$$T(z) = \frac{az+b}{cz+d},$$

w_0 a complex number and $r > 0$ then

$$|T(z) - w_0| = r$$

defines a circle or a line.

Problem 4. Prove this. $\square$

And now to the goal we are after.

Theorem 8. *Let S be the Möbius transformation*

$$S(z) = \frac{az+b}{cz+d}$$

and let

$$C = \{z : |z - w_0| = r\}$$

be the circle with center w_0 and radius r. Then the image

$$S[C] = \{S(z) : z \in C\}$$

is either a circle or a line.

Problem 5. Prove this. *Hint:* Here is one way to start. The image we are after is

$$S[C] = \{S(z) : |z - w_0| = r\}.$$

We will do a change of variable in this set. Let

$$T(z) = \frac{dz-b}{-cz+a}$$

Then use Theorem 3 to show that if we set $w = S(z)$, then

$$z = T(w).$$

Use this to show

$$S[C] = \{w : |T(w) - w_0| = r\}.$$

It should not be hard to finish from here. $\square$