Mathematics 552 Homework, February 3, 2020

Let U be an open subset of the complex plane U and let $f\colon U \to \mathbb{C}$ be a complex valued function defined on U. Then, in analogy with the definition in calculus, it is natural to define f to be ***differentiable*** at $z \in U$ if and only if the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

exits.

It is easy to check that many of the functions are are familiar with, such as polynomials, are differentiable at all points. Here is an example

Problem 1. Let $f(z) = z^4$. Use the Binomial Theorem to expand $(z+\Delta z)^4$ in the difference quotient

$$\frac{f(z+\Delta z) - f(z)}{\Delta z}$$

simplify the result and cancel the Δz out of the denominator to show that $f'(z) = 4z^3$. □

Problem 2. For a function that takes a bit more work let

$$f(z) = \frac{1}{z^2}$$

and show directly from the limit definition of $f'(z)$ that

$$f'(z) = \frac{-2}{z^3}$$

at all points where $z \neq 0$. □

Problem 3. Now let $f(z)$ be the function

$$f(z) = e^{az}$$

where a is a complex constant. Then

$$f(z) = 1 + (az) + \frac{(az)^2}{2!} + \frac{(az)^3}{3!} + \frac{(az)^4}{4!} + \frac{(az)^5}{5!} + \cdots$$

In this problem you will compute the derivative of $f(z)$ at $z = 0$. Verify that the difference quotient for f at $z = 0$ can be simplified as follows:

$$\begin{aligned}
\frac{f(0+\Delta z) - f(0)}{\Delta z} &= \frac{f(\Delta z) - 1}{\Delta z} \\
&= \frac{1}{\Delta z}\left(e^{a\Delta z} - 1\right) \\
&= \frac{1}{\Delta z}\left(\left(1 + (a\Delta z) + \frac{(a\Delta z)^2}{2!} + \frac{(a\Delta z)^3}{3!} + \frac{(a\Delta z)^4}{4!} + \cdots\right) - 1\right) \\
&= \frac{1}{\Delta z}\left((a\Delta z) + \frac{(a\Delta z)^2}{2!} + \frac{(a\Delta z)^3}{3!} + \frac{(a\Delta z)^4}{4!} + \cdots\right) \\
&= a + \frac{a^2\Delta z}{2!} + \frac{a^3(\Delta z)^2}{3!} + \frac{a^4(\Delta z)^3}{4!} + \cdots
\end{aligned}$$

And use this to show that

$$\begin{aligned} f'(0) &= \lim_{\Delta z \to} \frac{f(0+\Delta z) - f(0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(a + \frac{a^2 \Delta z}{2!} + \frac{a^3 (\Delta z)^2}{3!} + \frac{a^4 (\Delta z)^3}{4!} + \cdots \right) \\ &= a + 0 + 0 + 0 + \cdots \\ &= a. \end{aligned}$$

□

The main point of the previous problem can be summarized as

$$\lim_{\Delta z \to 0} \frac{e^{a\Delta z} - 1}{\Delta z} = a. \tag{1}$$

Problem 4. Let $f(z) = e^{az}$. Use the Limit (1) to show that

$$f'(z) = ae^{az}.$$

Hint: We know that the exponential function satisfies $e^{z_1 + z_2} = e^{z_1} e^{z_2}$. Use this to show

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \left(\frac{e^{a\Delta z} - 1}{\Delta z} \right) e^{az}.$$

and hopefully it is clear what to do from here. □

So we now have that for any complex constant that

$$\frac{d}{dz} e^{az} = ae^{az}.$$

Problem 5. Use this and

$$\sin(az) = \frac{e^{iaz} - e^{-iaz}}{2i}, \qquad \cos(az) = \frac{e^{iaz} + e^{-iaz}}{2}$$

to find formulas for the derivatives of $\sin(az)$ and $\cos(az)$. □