Mathematics 552 Homework, February 10, 2020

This homework is not to be collected, but is to give some more examples of finding the radius of convergence using the ratio. First we recall:

Theorem 1. *Let*

$$\sum_{n=0}^{\infty} a_n$$

be a series of complex numbers such that

$$\mathsf{ratio} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

exists. Then
(a) If $\mathsf{ratio} < 1$, *the series converges.*
(b) If $\mathsf{ratio} > 0$, *the series diverges.*
(c) If $\mathsf{ratio} = 1$, *the test fails and no conclusion can be made.* □

Problem 1*.* Find the radius of convergence of the series

$$f(x) = \sum_{n=0}^{\infty} n^3 2^n z^n.$$

Solution. Here we have

$$\mathsf{ratio} = \lim_{n\to\infty} \left| \frac{(n+1)^3 2^{n+1} z^{n+1}}{n^3 2^n z^n} \right| = \lim_{n\to\infty} \frac{(n+1)^3 2|z|}{n^3} = 2|z|.$$

Thus the series converges when $\mathsf{ratio} = 2|z| < 1$, that is when $|z| < 1/2$. Therefore radius of converges is $R = 1/2$ and the series converges at all points in the disk $\{z : |z| < 1\}$. □

Problem 2*.* Find the radius of convergence of the series

$$g(z) = \sum_{n=0}^{\infty} \frac{(z-i)^{3n+1}}{n10^n}.$$

Solution. Again we compute the ratio:

$$\mathsf{ratio} = \lim_{n\to\infty} \left| \frac{(z+i)^{3(n+1)+1}}{(n+1)10^{n+1}} \frac{n10^n}{(z-i)^{3n+1}} \right| = \lim_{n\to\infty} \frac{n|z+i|^3}{(n+1)10} = \frac{|z+i|^3}{10}.$$

Thus we have

$$\mathsf{ratio} = \frac{|z+i|^3}{10} < 1$$

when $|z+i|^3 < 10$, that is when $|z+i| < \sqrt[3]{10}$. Therefore the radius of convergence is $R = \sqrt[3]{10}$. And the series will converge for all points inside the circle with center $-i$ and radius $\sqrt[3]{10}$. □