

**Mathematics 552 Test 1**Name: Answer key**1.** Compute the following and put the answers in the form  $a + bi$ :

(a)  $(3 - 2i)(5 + 3i) = 21 - i$

(b)  $\frac{1 + 2i}{3 + 4i} = \frac{11}{25} + \frac{2}{25}i$

(c)  $(1 - i)^8$

*Solution:* There are several ways to do this. Maybe the most natural is to write in polar form:

$$(1 - i)^8 = \left(\sqrt{2}e^{-\pi i/4}\right)^8 = 2^{\frac{8}{2}}e^{-2\pi i} = 2^4(1) = 16.$$

A couple of you did the clever trick of repeated squaring:

$$(1 - i)^8 = ((1 - i)^2)^4 = (-2i)^4 = ((-2i)^2)^2 = (-4)^2 = 16.$$

And for yet another method use the general definition of  $z^\alpha = e^{\alpha \log(z)}$ :

$$(1 - i)^8 = e^{8 \log(1 - i)} = e^{8(\ln(\sqrt{2}) + (-\pi/4 + 2\pi n))} = e^{8(\frac{1}{2} \ln(2))} e^{(-2\pi + 16\pi n)} = 16(1) = 16.$$

And finally a few of you used Pascal's triangle to expand  $(1 - i)^8$ . This works, but involves way too much work. □

(d)  $\sqrt{-2 + 2i\sqrt{3}}$

*Solution:* Another one where the polar form is most natural:

$$\sqrt{-2 + i2\sqrt{3}} = \left(4e^{i(2\pi/3 + 2\pi n)}\right)^{\frac{1}{2}} = 2e^{i\pi/3 + n\pi} = \pm 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \pm(1 + i\sqrt{3})$$

□

(e)  $\log(-1 - i)$

*Solution:*

$$\log(-1 - i) = \ln(|-1 - i|) + i \arg(-1 - i) = \ln(\sqrt{2}) + i\left(\frac{5\pi}{4} + 2\pi n\right).$$

□**2.** State Euler's formula for  $e^{iz}$ . *Solution:*  $e^{iz} = \cos(z) + i \sin(z)$ **3.** Find all solutions to  $z^3 = -8$ .*Solution:* The solutions are  $z = (-8)^{\frac{1}{3}}$ .

$$\begin{aligned} (-8)^{\frac{1}{3}} &= \left(8e^{\pi i + 2n\pi i}\right)^{\frac{1}{3}} \\ &= 2e^{\frac{(2n+1)\pi i}{3}} \\ &= 2e^{\frac{\pi i}{3}}, 2e^{\pi i}, 2e^{\frac{5\pi i}{3}} \\ &= 1 + \sqrt{3}i, -2, 1 - \sqrt{3}i. \end{aligned}$$

□

4. Find the sum of the series

$$f(z) = \sum_{n=0}^{\infty} 5(-1)^n z^{3n} = 5 - 5z^3 + 5z^6 - 5z^9 + 5z^{12} - \dots$$

*Solution:* This is a geometric series therefore the sum is

$$f(z) = \frac{\text{first}}{1 - \text{ratio}} = \frac{5}{1 - (-z^3)} = \frac{5}{1 + z^3}$$

This converges when  $|\text{ratio}| = |-z^3| < 1$ . That is when  $|z| < 1$ . □

5. Give the series definitions of the following functions:  $e^z$  and  $\cos(z)$ .

*Solution.*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$

□

6. (a) Let  $f(z)$  be defined on the open set  $U$  and let  $z_0 \in U$ . Give the limit definition of what it means for  $f(z)$  to be differentiable at  $z_0$ .

*Solution:*  $f$  is differentiable at  $z_0$  if the limit

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

It is equally acceptable to write the limit in the forms:

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

□

(b) What is the definition of  $f(z)$  being ***analytic*** in the open set  $U$ .

*Solution:*  $f$  is analytic in  $U$  if and only if  $f'(z)$  exists at all points  $z \in U$ . □

7. (a) Let  $f(z) = u + iv$  in the open set  $U$ . State what it means for  $f(z)$  to satisfy the **Cauchy-Riemann equations**.

*Solution:* The Cauchy-Riemann equations are

$$u_x = v_y, \quad u_y = -v_x$$

or if you prefer not using the subscript notation for partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

□

(b) Let  $g(z) = (x^2 + y^2) + i(3xy + 1)$ . Explain why  $g(z)$  is not analytic.

*Solution:* It is a theorem that a function  $u + iv$  defined on an open subset of  $\mathbb{C}$  is analytic if and only if it satisfies the CR-equations on that set. In this case  $u = x^2 + y^2$  and  $v = 3xy + 1$ . Therefore

$$\begin{array}{ll} u_x = 2x & v_x = 3y \\ u_y = 2y & v_y = 3x \end{array}$$

and we see that  $u_x \neq v_y$  and  $u_y \neq -v_x$ . Therefore neither CR-equation holds. Thus  $g$  is not analytic. □

(c) Let  $f(z) = (x^2 - y^2 - 2x) + i(2xy - 2y)$ . Explain why  $f(z)$  is analytic on  $\mathbb{C}$  and give a formula for its derivative.

*Solution:* If  $f = u + iv$  satisfies the CR-equations it is analytic and its derivative is  $f' = u_x + iv_x$ . Here  $u = x^2 - y^2 - 2x$  and  $v = 2xy - 2y$  and

$$\begin{array}{l} u_x = v_y = 2x - 2 \\ u_y = -v_x = -2y \end{array}$$

Thus the CR-equations hold and therefore  $f$  is analytic on all of  $\mathbb{C}$ . Its derivative is

$$f'(z) = u_x + v_x i = 2x - 2 + 2y i.$$

Another way to have done this is note that if  $z = x + iy$ , then

$$z^2 - 2z = (x + iy)^2 - 2(x + iy) = (x^2 - y^2 - 2x) + (2xy - 2y)i = f(z).$$

Therefore  $f(z) = z^2 - 2z$  is a polynomial and therefore analytic and

$$f'(z) = 2z + 2 = (2x + 2) + 2y i$$

as before. □

(d) If  $f(z) = u + iv$  is analytic in the open set  $U$  show that  $v$  satisfies the equation

$$v_{xx} + v_{yy} = 0.$$

*Solution:* This uses the CR-equations:

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = -u_{xy} + u_{xy} = 0.$$

□

8. Find the center and radius of the circle of convergence of the following series

$$(a) g(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)^3 2^k}.$$

*Solution:* We use the ratio test:

$$\begin{aligned} \text{ratio} &= \lim_{k \rightarrow \infty} \left| \frac{(k+1)\text{-st term}}{k\text{-th term}} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{z^{k+1}}{((k+1)+2)^3 2^{k+1}} \cdot \frac{(k+2)^3 2^k}{z^k} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{(k+2)^3 z}{(k+3)^3 2} \right| \\ &= \frac{|z|}{2} \end{aligned}$$

The series converges when

$$\text{ratio} = \frac{|z|}{2} < 1$$

which is equivalent to  $|z| < 2$ . Thus

$$\text{Center} = 0,$$

$$\text{Radius} = 2.$$

□

$$(b) f(z) = \sum_{n=0}^{\infty} \frac{n(z+2i)^{2n+1}}{9^n}.$$

*Solution:* This time for convergence we want

$$\begin{aligned} \text{ratio} &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(z+2i)^{2(n+1)+1}}{9^{n+1}} \cdot \frac{9^n}{n(z+2i)^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(z+2i)^2}{9n} \right| \\ &= \frac{|z+2i|^2}{9} \\ &< 1. \end{aligned}$$

This is equivalent to  $|z+2i|^2 < 9$  which in turn is equivalent to  $|z+2i| < 3$ . Thus

$$\text{Center} = -2i$$

$$\text{Radius} = 3.$$

□

9. Find all solutions to  $\cos(z) = -\frac{5}{4}$ .

*Solution:* Using formula for  $\cos(z)$  in terms of the exponential we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = -\frac{5}{4}.$$

Multiply by  $4e^{-iz}$  to get

$$2(e^{iz})^2 + 2 = -5e^{iz}.$$

which can be rearranged as an equation that is quadratic in  $e^{-z}$ :

$$2(e^{iz})^2 + 5e^{iz} + 2 = 0.$$

By the quadratic formula

$$e^{iz} = \frac{-5 \pm \sqrt{25 - 4 \cdot 4}}{4} = \frac{-5 \pm \sqrt{9}}{4} = \frac{-5 \pm 3}{4} = -2, -\frac{1}{2}.$$

Solving  $e^{iz} = -2$  gives

$$z = \frac{1}{i} (\log(-2) = \ln(2) + i \arg(-2)) = -i \ln(2) + \pi + 2n\pi.$$

Solving  $e^{iz} = -\frac{1}{2}$  gives

$$z = \frac{1}{i} \log\left(\frac{-1}{2}\right) = i \ln(2) + \pi + 2n\pi.$$

Therefore the solutions are

$$z = (2n + 1)\pi \pm i \ln 2.$$

□