Math 552 Test 2, Answer key.

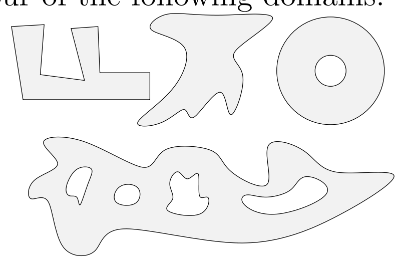
We have proven:

Theorem 1 (Cauchy Integral Theorem). *Let D be a bounded domain in $\mathbb{C}$ with nice boundary ∂D, which we transverse in the direction so that the interior of D is on the left. Let $f(z)$ be a function which is analytic in D and on the boundary ∂D. Then*

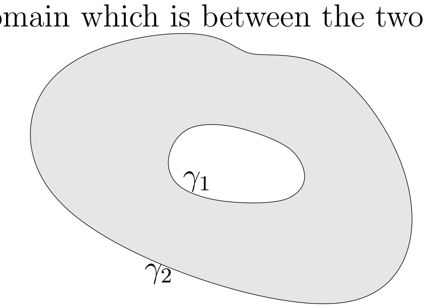
$$\int_{\partial D} f(z)\, dz = 0.$$

□

Note that this hold even when D is not simply connected. For example it holds for four of the following domains.

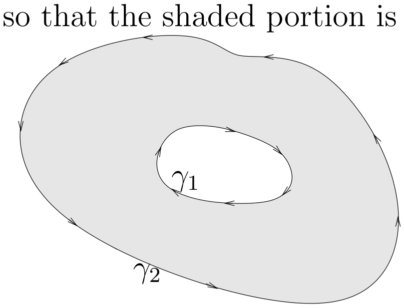


Problem 1. Let γ_1 and γ be two simple closed curves with γ_1 inside of γ_2 and D the domain which is between the two curves as shown.



(a) Sketch this domain on your exam and put arrows on γ_1 and γ_2 showing which way they are being traversed.

Solution. Move so that the shaded portion is on the left:



□

(b) Let $f(z)$ be a function that is analytic on γ_1, γ_2, and D. Use Cauchy's Integral Theorem to show

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Note that this is being graded in part for how it is written. So formulas with no English will be graded down. You should say why Cauchy's Theorem applies and why the integrals have have the signs that they do. □

Solution. If we orient the curves so that we move with the inside of the curve on the left (which is our convention) then the orientation of the outside curve, γ_2, agrees with its orientation as part of the boundary of D. On the other hand the orientation of the inside curve, γ_1, viewed as part of the boundary of D has the opposite orientation from our convention of moving so that the inside is on the left. Therefore by Cauchy's Integral Theorem

$$\begin{aligned} 0 &= \int_{\partial D} f(z)\,dz \\ &= \int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz \end{aligned}$$

which can be rearranged to give

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

as required. □

Remark. In this problem it is not true that the Cauchy integral implies

$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz = 0$$

because $f(z)$ need not be analytic inside of γ_1. An example is to let γ_1 and γ_2 be the circles $|z| = 1$ and $f(z) = 1/z$. Then

$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz = 2\pi i \neq 0.$$

□

Two more of our Theorems are

Theorem 2. *Let $f(z) = u + iv$ be defined in an open set U and such that the partial derivatives u_x, u_y, v_x, and v_y are continuous and satisfy the Cauchy-Riemann equations*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Then $f(z)$ is analytic in U. □

Theorem 3. *Let $f(z)$ be analytic in simply connected open set U. Then there is a function $F(z)$ with $F'(z) = f(z)$ in U. (That is an analytic function in a simply connected domain has an antiderivative in that domain.)* □

Recall that a real valued function h defined on an open subset U of $\mathbb{C}$ is ***harmonic*** if and only if

$$h_{xx} + h_{yy} = 0.$$

On Test 1 you proved the following

Proposition 4. *If h is harmonic on the open set U and we set*

$$u = h_x, \qquad v = -h_y$$

then $f(z) = u + iv$ satisfies the Cauchy-Riemann equations. □

We will now prove

Theorem 5. *Let h be harmonic in a simply connected open set D. Then there is an analytic function $H(z)$ with*

$$\operatorname{Re}(H(z)) = h(z)$$

in U. (That is in a simply connected domain every harmonic function is the real part of an analytic function.)

Problem 2. Prove this along the following lines

(a) Let

$$f(z) = h_x - ih_y$$

and explain why Proposition 4 together with Theorem 2 imply $f(z)$ is analytic in D.

Solution. By Proposition 4 the function satisfies the Cauchy-Riemann equations. By Theorem 2 this implies $f(z)$ is analytic. □

(b) Explain why $f(z)$ has an antiderivative in D. Call this antiderivative $F(z)$. That is $F'(z) = f(z)$ and write

$$F(z) = U + iV.$$

Solution. The domain D is given to be simply connected and $f(z)$ is analytic therefore by Theorem 3 there is an analytic function $F(z)$ difined on D with $F'(z) = f(z)$. □

(c) Use that the derivative of $F(z)$ can be computed by the formula

$$F'(z) = U_x + iV_x$$

and the Cauchy-Riemann equations to show

$$U_x = h_x, \qquad U_y = h_y$$

Solution. Using the given formula for $F'(z)$ and that $F'(z) = f(z)$ we have

$$F'(z) = U_x + iV_x = h_x - ih_y.$$

Comparing the real parts gives

$$U_x = h_x$$

which is half of what we want. Comparing the imaginary parts gives

$$V_x = -h_y.$$

Since $F(z)$ is analytic U and V satisfy the Cauchy-Riemann equations and thus

$$U_y = -V_x = -(-h_y) = h_y$$

and we done. □

(d) Explain why $U(z) = h(z) + c$ where c is constant and use this to finish the proof. □

Letting $\phi = U - h$ we have from part (c) that

$$\phi_x = U_x - h_x = 0, \qquad \phi_y = U_y - v_y$$

which implies ϕ is constant in D. (I.e. $\nabla\phi = (0,0)$ and we know from Math 241 that a function with zero gradient is constant.) Let $\phi = c$ a constant, then $U(z) - h(z) = c$ and thus $U(z) = h(z) + c$. This gives

$$h(z) = U(z) - c = \mathrm{Re}((U(z) - c) + iV(z)) = \mathrm{Re}(F(z) - c) = \mathrm{Re}(H(z))$$

where

$$H(z) = F(z) - c$$

is an analytic function.

Anther of our recent results is

Theorem 6. *Let $f(z)$ be analytic in the simply connected open set U and assume $f(z) \neq 0$ for all z in U. Then $f(z)$ has an analytic logarithm in U. That is there is an analytic function $g(z)$ in U such that $f(z) = e^{g(z)}$.* □

Also recall that we have defined $\cosh(z)$ and $\sinh(z)$ by

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \frac{e^z - e^{-z}}{2}.$$

Problem 3. Let $f(z)$ and $g(x)$ be analytic functions in on a simply connected open set U such that

$$f(z)^2 - g(z)^2 = 1$$

for all $z \in U$. Show that there is an analytic function $h(x)$ such that

$$f(z) = \cosh(h(z)), \qquad g(z) = \sinh(h(z)).$$

Hint: To start factor and rewrite the equation as

$$(f(z) + g(z))(f(z) - g(z)) = 1.$$

(a) Explain why the functions $f(z)+g(z)$ and $f(z)-g(z)$ never vanish on the set U.

Solution. As $(f(z) + g(z))(f(z) - g(z)) = 1 \neq 0$ neither of the factors $f(z) + g(z)$ or $f(z) - g(z)$ can vanish. □

(b) Use part (a) to explain why there are analytic functions $h_1(z)$ and $h_2(z)$ defined on U with

$$f(z) + g(z) = e^{h_1(z)}$$
$$f(z) - g(z) = e^{h_2(z)}.$$

Solution. This is a direct consequence of Theorem 6. The functions $f(z)+g(z)$ and $f(z)-g(z)$ do not vanish on the simply connected domain U and therefore they have analytic logarithms. So there are analytic functions $h_(z)$ and $h_2(z)$ such that

$$f(z) + g(z) = e^{h_1(z)}$$
$$f(z) - g(z) = e^{h_2(z)}.$$

□

(c) Show that $e^{h_1(z)+h_2(z)} = 1$ and therefore $h_1(z) + h_2(z) = 2n\pi i$ for some integer n.

Solution. We have

$$e^{h_1(z)+h_2(z)} = e^{h_1(z)}e^{h_2(z)} = (f(z) + g(z))(f(z) - g(z)) = 1$$

and this implies $h_1(z) + h_2(z) = 2n\pi i$ for some integer n. □

(d) Show that if $h(z) = h_1(z)$ then

$$f(z) + g(z) = e^{h(z)}$$
$$f(z) - g(z) = e^{-h(z)}.$$

Solution. We have

$$1 = (f(z) + g(z))(f(z) - g(z)) = e^{h(z)}e^{h_2(z)}.$$

Dividing by $e^{h(z)}$ gives

$$e^{h_2(z)} = \frac{1}{e^{h_1(z)}} = e^{-h(z)}.$$

Thus

$$f(z) + g(z) = e^{h_1(z)} = e^{h(z)}$$
$$f(z) - g(z) = e^{h_2(z)} = e^{-h(z)}$$

□

(e) Now finish the proof.

Solution. We can solve

$$f(z) + g(z) = e^{h(z)}$$
$$f(z) - g(z) = e^{-h(z)}.$$

for $f(z)$ and $g(z)$. To be explicit adding these equations gives

$$2f(z) = e^{h(z)} + e^{-h(z)}$$

and subtracting them gives

$$2g(z) = e^{h(z)} - e^{-h(z)}$$

Dividing by 2 gives

$$f(z) = \frac{e^{h(z)} + e^{-h(z)}}{2} = \cosh(h(z))$$
$$g(z) = \frac{e^{h(z)} - e^{-h(z)}}{2} = \sinh(h(z))$$

□

Problem 4. Find the radius of convergence of the following series:

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k z^k}{k+5}$.

(b) $\displaystyle\sum_{n=0}^{\infty} n^3 3^n (z+1+2i)^{2n}$. □

Solution. We will use the ratio test on both of these series. For the first one

$$\begin{aligned}\mathsf{ratio} &= \lim_{k\to\infty}\left|\left(\frac{2^{k+1}z^{k+1}}{k+1+5}\right)\left(\frac{k+5}{2^k z^k}\right)\right| \\ &= \lim_{k\to\infty}\left|\frac{2z(k+5)}{k+6}\right| \\ &= 2|z|\end{aligned}$$

The series converges when $\mathsf{ratio} < 1$ so the radius of convergence is

$$R = \frac{1}{2}.$$

For the second series

$$\begin{aligned}\mathsf{ratio} &= \lim_{n\to\infty}\left|\frac{(n+1)^3 3^{n+1}(z+1+2i)^{2(n+1)}}{n^3 3^n (z+1+2i)^{2n}}\right| \\ &= \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^3 3(z+1+2i)^2\right| \\ &= 3|z+1+2i|^2\end{aligned}$$

Thus $\mathsf{ratio} < 1$ when

$$|z+1+2i|^2 < \frac{1}{3}.$$

Taking square roots gives

$$|z+1+2i| < \frac{1}{\sqrt{3}}$$

and therefore the radius of convergence is

$$R = \frac{1}{\sqrt{3}}$$

□

Using that

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} = 1 + w + w^2 + w^3 + w^4 + \cdots \tag{1}$$

(which converges for $|w| < 1$) we have found power series expansions of some rational functions. For example let us find the expansion of

$$\frac{1}{5-z}$$

about $z = 3$, that is in powers of $(z-3)$.

$$\begin{aligned}
\frac{1}{5-z} &= \frac{1}{5-(z-3+3)} \\
&= \frac{1}{2-(z-3)} \\
&= \frac{1}{2}\left(\frac{1}{1-\left(\frac{z-3}{2}\right)}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z-3}{2}\right)^k \qquad \text{Using (1) with } w = \frac{z-3}{2} \\
&= \sum_{k=0}^{\infty}\frac{(z-3)^k}{2^{k+1}}.
\end{aligned}$$

This converges when

$$|w| = \left|\frac{z-3}{2}\right| < 1$$

that is when $|z-3| < 2$. Therefore the radius of convergence is 2.

Problem 5. Find the power series expansion of

$$f(z) = \frac{1}{10-z}$$

about the points $z = 1 + 2i$ and give its radius of convergence. $\square$

Solution. To simplify notation let $a = 1 + 2i$. Then

$$\begin{aligned} f(z) &= \frac{1}{10 - z} \\ &= \frac{1}{(10 - a) - (z - a)} \\ &= \frac{1}{10 - a}\left(\frac{1}{1 - \dfrac{z - a}{10 - a}}\right) \\ &= \frac{1}{10 - a}\sum_{k=0}^{\infty}\left(\frac{z - a}{10 - a}\right)^k \qquad \text{Using (1) with } w = \frac{z - a}{10 - a} \\ &= \sum_{k=0}^{\infty}\frac{(z - a)^k}{(10 - a)^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(z - 1 - 2i)^k}{(9 - 2i)^{k+1}} \end{aligned}$$

This is a geometric series with ratio

$$r = \frac{(z - a)^k}{(10 - a)^{k+1}} = \frac{z - 1 - 2i}{9 - 2i}$$

so the series converges when

$$|r| = \left|\frac{z - 1 - 2i}{9 - 2i}\right| = \frac{|z - 1 - 2i|}{\sqrt{85}} < 1.$$

Therefore

$$R = \sqrt{85}$$

is the radius of convergence. □

Problem 6. Find the power series expansion of

$$\frac{1}{w - z}$$

about the complex number $z = c \neq w$ and give its radius of convergence. □

Solution. This is very like the previous problem:

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-c)-(z-c)}\\ &= \frac{1}{w-c}\left(\frac{1}{1-\left(\frac{z-c}{w-c}\right)}\right)\\ &= \frac{1}{w-c}\sum_{k=0}^{\infty}\left(\frac{z-c}{w-c}\right)^k\\ &= \sum_{k=0}^{\infty}\frac{(z-c)^k}{(w-c)^{k+1}}\end{aligned}$$

This is a geometric series with ratio

$$r = \frac{z-c}{w-c}.$$

Thus $|r| < 1$ if and only if $|z-c| < |w-c|$. Thus

$$R = |w-c|$$

is the radius of convergence. □

Anther big result we have done since the last test is

Theorem 7 (Liouville's theorem)**.** *A bounded entire function is constant.* □

Problem 7. Let $f(z)$ be an entire function with $|f(z)| \geq 1$ for all z. Use Liouville's Theorem to show $f(z)$ is constant. □

Solution. Since $|f(z)| \geq 1$ we have $f(z) \neq 0$ for all z. Therefore

$$g(z) = \frac{1}{f(z)}$$

is also entire (that is analytic at all points of $\mathbb{C}$). Since $|f(z)| \geq 1$ the function $g(z)$ has the bound

$$|g(z)| = \frac{1}{|f(z)|} \leq 1.$$

Therefore $g(z)$ is a bounded entire function and Liouville tells us it is constant. Therefore

$$f(z) = \frac{1}{g(z)}$$

is also constant. □

Problem 8. Let $f(z)$ and $g(z)$ be two entire functions such that

$$|f(z)| \geq 1 + |g(z)|$$

for all $z \in \mathbb{C}$. Show $f(z)$ and $g(z)$ are both constant. □

Solution. First note

$$|f(z)| \geq 1 + |g(z)| \geq 1$$

and therefore $f(z)$ is constant by Problem 7. Let $f(z) = c$. Then

$$|c| = |f(z)| \geq 1 + |g(z)|$$

which can be rearranged as

$$|g(z)| \leq |c| - 1$$

whence $g(z)$ is also a bounded entire function and thus also constant.

□