# Math 552 Test 3.

**Problem** 1. (a) Find the radius of convergence of the Taylor expansion of

$$f(z) = \frac{\sin(z^3 + 42)}{z^2 - 2z + 5}$$

about the point $z_0 = -3 + 6i$.

(b) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n^2+1)(z-1+2i)^{3n+1}}{5(27)^n}$$

□

**Solution.** (a) The singularities of are where $z^2 - 2z + 5 = 0$. Solve this gives

$$z = 1 \pm 2i.$$

The radius of convergence of the Taylor about $z_0$ is the distance of $z_0$ to the nearest of the singularities. These distances are

$$|z_0 - (1+2i)| = |-4+4i| = 4\sqrt{2}$$
$$|z_0 - (1-2i)| = |-4+8i| = 4\sqrt{5}.$$

Therefore the radius of convergence is

$$R = 4\sqrt{2}.$$

(b) We use the ratio test

$$\begin{aligned}
\text{ratio} &= \lim_{n\to\infty} \left| \frac{(n+1)\text{-st term}}{n\text{-th term}} \right| \\
&= \lim_{n\to\infty} \left| \left( \frac{((n+1)^2+1)(z-1+2i)^{3(n+1)+1}}{5(27)^{n+1}} \right) \left( \frac{5(27)^n}{(n^2+1)(z-1+2i)^{3n+1}} \right) \right| \\
&\lim_{n\to\infty} \left( \frac{(n+1)^2+1}{n^2+1} \right) \left( \frac{|z-1+2i|^3}{27} \right) \\
&= 1 \frac{|z-1+2i|^3}{27} \\
&= \frac{|z-1+2i|^3}{27}
\end{aligned}$$

The criterion for convergence is

$$\text{ratio} = \frac{|z-1+2i|^3}{27} < 1$$

that is

$$|z-1+2i|^3 < 27$$

which is equivalent to

$$|z - 1 + 21| < \sqrt[3]{27} = 3.$$

Thus the radius of convergence is

$$R = 3.$$ □

Many of the following problems are based on

**Theorem 1** (The Residue Theorem)**.** *Let $f(z)$ be analytic on a simple closed curve $\gamma$ and also analytic inside of $\gamma$ except at a finite number of isolated singularities $z_1, z_2, \ldots, z_m$. Then*

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f, z_k) \\ &= 2\pi(\text{sum of residues of } f(z) \text{ at singularities inside of } \gamma)\end{aligned}$$

□

We have a nice formula for the residues of $f(z)$ in some special cases.

**Theorem 2** (Formula for Residues at Simple Poles)**.** *Let $f(z)$ be of the form*

$$f(z) = \frac{g(z)}{h(z)}$$

*where $g(z)$ and $h(z)$ are analytic in a disk about $z_0$ and*

$$h(z_0) = 0, \qquad h'(z_0) \neq 0.$$

*Then the residue of $f(z)$ at $z_0$ is*

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

**Problem** 2. Compute $\operatorname{Res}(f, z_0)$ in the following cases:

(a) $f(z) = \dfrac{e^{z^2}}{z^2+4}$ and $z_0 = 2i$.

(b) $f(z) = \tan(z)$ and $z_0 = \dfrac{\pi}{2}$

(c) $f(z) = \dfrac{g(z)}{z-a}$ where $g(z)$ is analytic and $z_0 = a$. □

**Solution.** (a) In this case $g(z) = e^{z^2}$, $h(z) = z^2$, and $h'(z) = 2z$. Thus

$$g(z_0) = e^{(2i)^2} = e^{-4}, \qquad h'(z_0) = 2(2i) = 4i.$$

which gives the residue as

$$\operatorname{Res}(f, 2i) = \frac{g(2i)}{h'(2i)} = \frac{e^{-4}}{4i} = \frac{-i}{4e^4}.$$

(b) Write $\tan(z)$ as $\tan(z) = \dfrac{\sin(z)}{\cos(z)} = \dfrac{g(z)}{h(z)}$. Then $h'(z) = (\cos(z))' = -\sin(z)$ and

$$g(z_0) = \sin(\pi/2) = 1, \qquad h'(z_0) = -\sin(\pi/2) = -1.$$

So

$$\operatorname{Res}(f, z_0) = \frac{g(\pi/2)}{h'(\pi/2)} = \frac{1}{-1} = -1.$$

(c) When $f(z) = \frac{g(z)}{z-a}$ the denominator is $h(z) = z - a$ and $h'(z) = 1$. Thus the residue is

$$\operatorname{Res}(f, a) = \frac{g(a)}{1} = g(a).$$

Note that in this case if $\gamma$ is a simple closed curve with $a$ on the inside of $\gamma$ the Residue Theorem gives

$$\int_\gamma \frac{g(z)}{z-a}\, dz = 2\pi i \operatorname{Res}\left(\frac{g(z)}{z-a}, a\right) = 2\pi g(a)$$

which shows that the Residue Theorem implies the Cauchy Integral Formula. □

**Problem** 3. Compute the following

(a) $\displaystyle\int_{|z-5|=2} \frac{e^{z^2}}{z^2+4}\, dz.$

(b) $\displaystyle\int_{|z-3i|=2} \frac{e^{z^2}}{z^2+4}\, dz.$

**Solution.** (a) The function $f(z) = \dfrac{e^{z^2}}{z^2+4}$ has no singularites inside of the circle $|z-5| = 2$ and thus is analytic inside the circle. Therefore by the Cauchy Integral Theorem

$$\int_{|z-5|=2} \frac{e^{z^2}}{z^2+4}\, dz = 0.$$

(b) This time $f(z) = \dfrac{e^{z^2}}{z^2+4}$ has the one singularity $z = 2i$ inside of the circle $|z - 3i| = 2$. We computed the residue of $f(z)$ at $z = 2i$ in Problem 2 (a) and the Residue Theorem gives

$$\int_{|z-3i|=2} \frac{e^{z^2}}{z^2+4}\, dz = 2\pi i \operatorname{Res}(f(z), 2i) = 2\pi i \frac{e^{-4}}{4i} = \frac{\pi}{2e^4}.$$

□

**Problem** 4. For $R > 0$ let $\gamma_R$ be the upper half of the circle $|z| = R$ and $\sigma_r$ the line segment on the real axis between $-R$ and $R$. Let $C_R$ be union of $\gamma_R$ and $\sigma_R$, which is a closed curve and which we orient so that, as usual, we move along the curve so that inside of the curve on the left in Figure 1.

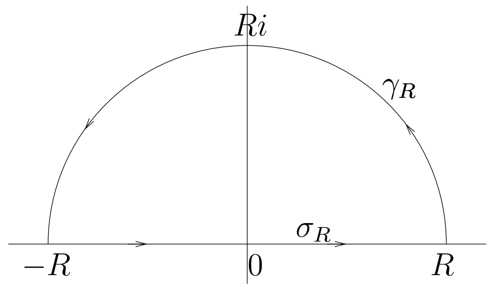


FIGURE 1. The curve $\gamma_R$ is the top half of the circle $|z| = R$ and $\sigma_R$ diameter of this circle between $-R$ and $R$. Then $C_R$ is the union of these two curve oriented as shown.

(a) What is the length of $\gamma_R$?

Now let $a$ and $b$ be positive real numbers with $a \neq b$ and let

$$f(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}.$$

(b) What are the singularities of $f(z)$?
(c) Compute the resides of $f(z)$ at its singularities.
(d) Assuming that $R$ is very large use the Residue Theorem to compute

$$\int_{C_R} f(z)\, dz.$$

□

**Solution.** (a) The curve $\gamma_R$ is half of a circle of radius $R$ and therefore

$$\text{Length}(\gamma_R) = \frac{1}{2}(2\pi R) = \pi R.$$

(b) The singularities of $f(z)$ are where $(z^2 + a^2)(z^2 + b^2) = 0$, that is the singularities are

$$ai, \quad -ai, \qquad bi, \qquad -b1.$$

(c) The residues at the singularities are

$$\begin{aligned}
\text{Res}(f, ai) &= \frac{-a}{2i(-a^2+b^2)} \\
\text{Res}(f, -ai) &= \frac{a}{2i(-a^2+b^2)} \\
\text{Res}(f, bi) &= \frac{-b}{2i(a^2-b^2)} \\
\text{Res}(f, -bi) &= \frac{b}{2i(a^2-b^2)}
\end{aligned}$$

(d) If $R$ is large, then $R > a$ and $R > b$. Then $C_R$ has just the two singularities $ai$ and $bi$ in its interior. So by the Residue Theorem we have

$$\begin{aligned}
\int_{C^R} f(z)\,dz &= 2\pi i\,(\text{Res}(f, ai) + \text{Res}(f, bi)) \\
&= 2\pi i \left( \frac{-a}{2i(-a^2+b^2)} + \frac{-b}{2i(a^2-b^2)} \right) \\
&= \pi \left( \frac{-a}{(-a^2+b^2)} + \frac{-b}{(a^2-b^2)} \right) \\
&= \pi \left( \frac{a-b}{a^2-b^2} \right) \\
&= \pi \left( \frac{a-b}{(a-b)(a+b)} \right) \\
&= \frac{\pi}{a+b}.
\end{aligned}$$

□

Recall our basic estimate for the size of complex line integrals:

**Proposition 3.** *Let $f(z)$ be continuous on a curve $\gamma$ and assume that for a positive constant $M$*

$$|f(z)| \leq M.$$

*Then*

$$\left| \int_\gamma f(z)\,dz \right| \leq M\,\text{Length}(\gamma)$$

□

**Problem** 5. Recall that the reverse triangle inequality for complex numbers is

$$|z_2 + z_1| \geq |z_2| - |z_1|.$$

(a) Use this to show that if $|z| = R$ and $a$ and $b$ are positive real numbers with $a, b < R$, then

$$|(z^2 + a^2)||(z^2 + b^2)| \geq (R^2 - a^2)(R^2 - b^2)$$

and therefore

$$\left| \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \right| \leq \frac{R^2}{(R^2 - a^2)(R^2 - b^2)}.$$

(b) Let $f(z) = \dfrac{z^2}{(z^2 + a^2)(z^2 + b^2)}$ be the function of Problem 4 and also $\gamma_R$ and $\sigma_R$ as in Problem 4. Use part (a) of the current problem and Theorem 3 to show that when $R > a, b$ the inequality

$$\left| \int_{\gamma_R} f(z)\, dz \right| \leq \frac{\pi R^3}{(R^2 - a^2)(R^2 - b^2)}$$

holds.

(c) Show

$$\lim_{R \to \infty} \int_{\gamma_R} f(z)\, dz = 0.$$

(d) Explain why

$$\int_{\sigma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx.$$

(e) Justify the following

$$\lim_{R \to \infty} \int_{C_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}\, dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\, dx$$

(f) What is $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\, dx$ ?

**Solution.** (a) If $|z| = R$ then by the reverse triangle inequality we have

$$|z^2 + a^2| \geq |z|^2 - |a|^2 = R^2 - a^2 \qquad (\text{as } a \text{ is real so } |a|^2 = a^2)$$
$$|z^2 + b^2| \geq |z|^2 - |b|^2 = R^2 - b^2 \qquad (\text{as } b \text{ is real so } |b|^2 = b^2)$$

As $R > a, b$ the numbers $R^2 - a^2$ and $R^2 - b^2$ are positive so we can multiply the inequalities to get

$$|z^2 + a^2||z^2 + b^2| \geq (R^2 - a^2)(R^2 - b^2)$$

and taking the reciprocal gives

$$\frac{1}{|z^2 + a^2||z^2 + b^2|} \leq \frac{1}{(R^2 - a^2)(R^2 - b^2)}$$

Now multiply by $|z|^2$ and use $|z| = R$

$$\frac{|z|^2}{|z^2+a^2||z^2+b^2|} \le \frac{|z|^2}{(R^2-a^2)(R^2-b^2)} = \frac{R^2}{(R^2-a^2)(R^2-b^2)}.$$

(b) From part (a) we have

$$|f(z)| \le \frac{R^2}{(R^2-a^2)(R^2-b^2)}.$$

Using $M = \dfrac{R^2}{(R^2-a^2)(R^2-b^2)}$ in Proposition 3 we have

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \frac{R^2}{(R^2-a^2)(R^2-b^2)}\,\text{Length}(\gamma_R) = \frac{\pi R^3}{(R^2-a^2)(R^2-b^2)}$$

as $\text{Length}(\gamma_R) = \pi R$ by Problem 4 (a).

(c) Using what we have just shown

$$\lim_{R\to\infty}\left|\int_{\gamma_R} f(z)\,dz\right| \le \lim_{R\to\infty}\frac{\pi R^3}{(R^2-a^2)(R^2-b^2)} = 0.$$

This implies

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = 0.$$

For those wanting more detail on the limit divide top and bottom of the fraction by $R^4$ to get

$$\begin{aligned}\lim_{R\to\infty}\frac{\pi R^3}{(R^2-a^2)(R^2-b^2)} &= \lim_{R\to\infty}\frac{(\pi/R)}{(1-(a/R)^2)(1-(b/R)^2)}\\ &= \frac{0}{(1+0^2)(1+0^2)}\\ &= 0.\end{aligned}$$

(d) The segment $\sigma_R$ is the part of real axis between $-R$ and $R$. This is parameterized by $z = x + 0i = x$ with $-R \le x \le R$. Then $dz = dx$. Thus

$$\int_{\sigma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx = \int_{-R}^{R}\frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx.$$

(e) There is not much to do here:

$$\lim_{R\to\infty}\int_{C_R}\frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = \int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$$

by the definition of improper integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$.

(f) This follows from the previous two parts of this problem. To simplify notation let $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$

$$\begin{aligned}\lim_{R\to\infty}\int_{C_R} f(z)\,dz &= \lim_{R\to\infty}\left(\int_{\gamma_R} f(z)\,dz + \int_{\sigma_R} f(z)\,dz\right)\\ &= 0 + \int_{-\infty}^{\infty} f(x)\,dx\\ &= \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx\end{aligned}$$

(g) For large $R$ we have seen by Problem 4 (b) that

$$\int_{D_R} f(z)\,dz = \frac{\pi}{a+b}.$$

and thus $\int_{C_R}$ is constant as a function of $R$. Therefore

$$\begin{aligned}\frac{\pi}{a+b} &= \lim_{R\to\infty}\int_{C_R} f(z)\,dz\\ &= \int_{-\infty}^{\infty} f(x)\,dx\\ &= \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx\end{aligned}$$

as required.

□

**Problem** 6. Recall that the analytic function $f(z)$ has a zero of order $m$ at $z_0$ if and only if

$$f(z) = (z-z_0)^m f_1(z)$$

where $f_1(z)$ is analytic and $f_1(z_0) \neq 0$.

(a) Show

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{f_1'(z)}{f_1(z)}$$

(b) What is the residue of the function $\dfrac{f'(z)}{f(z)}$ at $z = z_0$ ?

(c) Let $\gamma$ be a simple closed curve and assume that $f(z)$ is analytic on and inside of $\gamma$. Also assume that $f(z) \neq 0$ on $\gamma$ (but $f(z)$ may have zeros inside of $\gamma$.) Use the Residue Theorem to explain why

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{number of zeros of } f(z) \text{ inside } \gamma \text{ counted with multiplicity.}$$

To be more explicit let $z_1, z_2, \ldots, z_k$ be the zeros of $f(z)$ inside of $\gamma$ and let $m_j$ be the order of $z_j$. Then you are to show

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = m_1 + m_2 + \cdots + m_k.$$

□

**Solution.** (a) This is just a plug and chug problem. If $f(z) = (z - z_0)^m f_1(z)$ then

$$f'(z) = m(z - z_0)^{m-1} f_1(z) + (z - z_0)^m f_1'(z)$$

Then dividing by $f(z)$ gives

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m(z - z_0)^{m-1} f_1(z)}{(z - z_0)^m f_1(z)} + \frac{(z - z_0)^m f_1'(z)}{(z - z_0)^m f_1(z)} \\ &= \frac{m}{(z - z_0)} + \frac{f_1'(z)}{f_1(z)}.\end{aligned}$$

(b) As $f_1(z_0) \neq 0$ the function $\frac{f_1'(z)}{f_1(z)}$ is analytic around $z_0$ and therefore has a Taylor expansion about $z_0$

$$\frac{f_1'(z)}{f_1(z)} = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Using this in our formula of part (a) gives

$$\frac{f'(z)}{f(z)} = \frac{m}{(z - z_0)} + \sum_{n=0}^{\infty} c_n (z - z_0)^n = m(z - z_0)^{-1} + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

as the Laurent expansion of $f'(z)/f(z)$ about $z = z_0$. By definition $\operatorname{Res}(f'/f, z_0)$ is the coefficient of $(z - z_0)^{-1}$ in the Laurent and therefore

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = m = \text{ Order of the zero of } f(z) \text{ at } z_0.$$

(c) Let $z_1, z_2, \ldots, z_k$ be the zeros of $f$ inside of $\gamma$ and let $m_j$ be the order of the zero of $f(z)$ at $z_j$. Then we have just seen that

$$\operatorname{Res}\left(\frac{f'}{f}, z_j\right) = m_j.$$

Therefore by the Residue Theorem

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)}\,dz &= 2\pi i \times \left(\text{Sum of residues of } \frac{f'}{f} \text{ inside } \gamma\right)\\ &= 2\pi i\left(\operatorname{Res}\left(\frac{f'}{f}, z_1\right) + \operatorname{Res}\left(\frac{f'}{f}, z_2\right) + \cdots + \operatorname{Res}\left(\frac{f'}{f}, z_k\right)\right)\\ &= 2\pi i\,(m_1 + m_2 + \cdots + m_k)\\ &= 2\pi i\,(\text{number of zeros of } f \text{ inside } \gamma \text{ counted with multiplicity})\end{aligned}$$

□

**Definition 4.** Let $f(z)$ have an isolated singularity at $z = z_0$ and $m$ a positive integer. Then $z = z_0$ is a ***pole of order*** $m$ of $f(z)$ if and only if the Laurent series of $f(z)$ is of the form

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{-m+1}} + \cdots = \sum_{n=-m}^{\infty} c_n (z-z_0)^n$$

with $c_{-m} \neq 0$. □

Thus if $f(z)$ has a pole of order 3 at $z_0$, then its Laurent expansion starts off as

$$f(z) = \frac{c_{-3}}{(z-z_0)^3} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

with $c_{-3} \neq 0$.

**Problem** 7. Let $f(z)$ be of the form

$$f(z) = \frac{g(z)}{h(z)}$$

where $g(z)$ and $h(z)$ are analytic in disk about $z_0$ and so that $g(z_0) \neq 0$ and $h(z)$ has a zero of order $m$ at $z_0$. Show that $f(z)$ has a pole of order $m$ at $z_0$. *Hint:* As $h(z)$ has a zero of order $m$ at $z_0$ you can write

$$h(z) = (z-z_0)^m h_1(z)$$

where $h_1(z)$ is analytic and $h_1(z_0) \neq 0$. Then

$$f(z) = \frac{g(z)}{(z-z_0)^m h_1(z)} = \frac{1}{(z-z_0)^m}\left(\frac{g(z)}{h_1(z)}\right) = \frac{f_1(z)}{(z-z_0)^m}$$

where

$$f_1(z) = \frac{g(z)}{h_1(z)}.$$

Explain why $f_1(z)$ is analytic in a disk about $z_0$ and why $f(z_0) \neq 0$. Use that $f_1(z)$ has a Taylor expansion about $z_0$ to finish the proof. □

**Solution.** With $f_1$ as in the hint we see that $f_1(z)$ is analytic near $z_0$ as $h_1(z_0) \neq 0$. Therefore $f_1(z)$ has a Taylor expansion about $z_0$:

$$f_1(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

and

$$c_0 = f_1(z_0) = \frac{g(z_0)}{h_1(z_0)} \neq 0$$

as $g(z_0) \neq 0$. Therefore

$$\begin{aligned} \frac{f_1(z)}{(z-z_0)^m} &= (z-z_0)^{-m} \sum_{k=0}^{\infty} c_k (z-z_0)^k \\ &= \sum_{k=0}^{\infty} c_k (z-z_0)^{k-m} \\ &= \sum_{n=-m} c_{n+m} (z-z_0)^n \end{aligned}$$

where we have done the change of variable $n - k - m$ is the sum. This Laurent series starts with the term

$$c_{-m+m}(z-z_0)^{-m} = c_0 (z-z_0)^{-m} \neq 0 \qquad \text{as } c_0 \neq 0.$$

Therefore this is the lowest order nonzero term in the Laurent expansion and therefore $f(z)$ has a pole of order $m$ at $z_0$. $\square$