Math 552, Final

- *This is due on Monday, May 3 by midnight. It should be submitted via Blackboard as a pdf document and should have your name on the first page.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *You will be graded in part on writing proofs up correctly. In particular you can lose points with answers that are all formulas and equations without any English.*

We started on the class reviewing some basic algebra. For the purposes of this class probably the most important topic review has geometric series. The sum of a finite geometric series is

$$\sum_{k=0}^{n} ar^k = a + ar + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r} = \frac{\mathsf{first} - \mathsf{next}}{1 - \mathsf{ratio}}$$

which holds for any complex numbers a and $r \neq 0$.

We latter did a lot with infinite geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} = \frac{\mathsf{first}}{1 - \mathsf{ratio}}.$$

which holds for all complex numbers a and $|r| < 1$.

Problem 1. (10 points)

(a) Find the sum of $\displaystyle\sum_{k=2}^{30} z^{2k+1}$.

(b) Show that if $\operatorname{Re} z < 0$, then $|e^z| < 1$ and find the sum of $\displaystyle\sum_{k=0}^{\infty} e^{kz}$.

Anther of the first things we did was to look for ways to extent the definitions of familiar functions $f(x)$ of a real variable, to function $f(z)$ for a complex variable. For polynomials and rational functions it is not hard to do this. For the functions $\sin(x)$, $\cos(x)$, and e^x Euler used the series expansions of these functions to extend them to the complex plane. In particular

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \end{aligned}$$

We then (still following what Euler did) used these series to show

$$\begin{aligned} e^{z+w} &= e^z e^w \\ e^{iz} &= \cos(z) + i\sin(z) \\ \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

We also defined $\arg(z)$, $\operatorname{Arg}(z)$, $\log(z)$ and $\operatorname{Log}(z)$.

Problem 2. (30 points) Compute the following:

(a) $(1+\sqrt{3}\,i)^{12}$ write answer in the form $x+iy$.
(b) $e^{3+\pi i}$ write the answer in the from $x+iy$.
(c) $\operatorname{Log}(-7-7i)$ write the answer in the from $x+iy$.
(d) $\operatorname{Arg}(\left(\sqrt{3}+i\right)^{17})$.
(e) The solution to $(3-4i)z = 11+2i$ write the answer in the from $x+iy$.
(f) The solution to $\dfrac{2z+6i}{z-1} = 3-i$ write the answer in the from $x+iy$ □

Problem 3. (15 points) Find all solutions to

$$\cos(3z) = 2.$$

Write your answers in the form $x = iy$. □

Problem 4. (10 points) Let a and z be complex numbers with $|a| \neq 1$ and $|z| = 1$. Show that

$$w = \frac{z-a}{\bar{a}z - 1}$$

satisfies $|w| = 1$. □

We then defined a complex valued function $f(z)$ defined on an open subset, U of $\mathbb{C}$ to be ***analytic*** if and only if its complex derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists at all points of U and then showed that our basic functions form calculus as expended to $\mathbb{C}$ by Euler's method of using power series are all analytic. In particular all polynomials, rational functions, the trigonometric functions, e^z, $\mathrm{Log}(z)$, and functions constructed from them by addition, multiplication, division, and function composition are all analytic where defined. As an example the function

$$f(z) = \frac{\cos(e^{3z}) - 9\sin(z^{42} - 3z)}{2 + e^z \sin(z^3 + 1)}$$

is analytic all all points where the denominator does not vanish.

A slightly more complicated example is complex powers. Let α be a complex number then for $z \neq 0$ we defined

$$z^\alpha = e^{\alpha \log(z)}$$

which is multivalued when α is not an integer. This is not really surprising as we are use to numbers having having two square roots, which is just to say that $z^{\frac{1}{2}}$ is two valued. When α is not a rational number, then z^α has infinitely many values. If we wanted just one value, we defined the ***principle value*** of z^α as

$$\text{Principle value of } z^\alpha = z^{\alpha \,\mathrm{Log}(z)}$$

where $\mathrm{Log}(z)$ is the principle value of $\log(z)$. With proper care about domains all of these functions are also analytic.

Problem 5. (15 points)

(a) Find all values of 1^i and write the answers in the from $x + iy$.

(b) Find all four values of $(-64)^{\frac{1}{4}}$ and write the answers in the form $x + iy$. □

We recall a fact from vector analysis.

Proposition 1. *Let $u(x, y)$ be a function on a connected open subset U of $\mathbb{R}^2$ with continuous first partial derivatives. Then if*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

on all of U, then $u = c$ for some constant c. □

Related to this is the important

Theorem 2 (Cauchy-Reimann Equations). *Let $f = u + iv$ be defined on the connected open subset U of $\mathbb{C}$ such that u and v have continuous first partial derivatives. Then f is analytic if and only if u and v satisfy the Cauchy-Riemann equations (and I am not stating the CR-equations as you all know them from memory).* □

Problem 6. (10 points) Let $f = u + iv$ be analytic in the connected open set and assume u and v satisfy

$$u^2 + 3v^2 = 4.$$

Show f is constant. □

The CR equations were then used to prove the Cauchy Integral Theorem which we stated as

$$\int_{\partial D} f(z)\, dz = 0$$

when $f(z)$ is analytic on D and ∂D and ∂D is nice. This was used to prove the Cauchy Integral Formula for $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z}\, dw.$$

and also a formula for the derivative

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z)^2}\, dw.$$

This, along with our basic estimate for integrals

Theorem 3 (Basic Integral Bound). *Let $f(z)$ be continuous along the curve γ and have $|f(z)| \le M$ along γ. Then*

$$\left| \int_\gamma f(z)\, dz \right| \le M \operatorname{Length}(\gamma).$$

was used to prove Liouville's theorem that an entire function $f(z)$ with $|f(z)| \le M$ for some constant M is constant. (That is a bounded entire function is constant.)

As a review of these ideas let us prove a generalization of Liouville's Theorem. Recall that one of the things we proved using the Cauchy Integral Formula is that if $f(z)$ is analytic in the disk $|z - z_0| \le R$ then $f(z)$ has a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

and we have two formulas for the coefficients one in terms of derivatives:

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

and one in terms of integrals:

$$c_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

for an r with $0 < r < R$.

Theorem 4. *Let $f(z)$ be an entire function that satisfies*

$$|f(z)| \le A + B|z| \tag{1}$$

for some positive constants. Then $f(z)$ is a linear function

$$f(z) = c_0 + c_1 z$$

for some complex constants.

Problem 7. (25 points) Prove this as follows. As $f(z)$ is entire, it has a series expansion about $z = 0$

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_2 z^3 + \cdots = \sum_{n=0}^{\infty} c_n z^n.$$

To prove the result we just need to prove

$$c_n = 0 \quad \text{for all} \quad n \ge 2$$

for then the series expansion will reduce to just $f(z) = c_0 + c_1 z$. Because $f(z)$ is entire its radius of convergence is infinite and therefore we have that the coefficients in the series are given by

$$c_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^n}\, dz$$

and this holds for any $r > 0$. To make this look a little more like some of the notation above let

$$\gamma_r = \text{the circle of radius } r \text{ defined by } |z| = r$$

and then our formula for c_n becomes

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^n}\, dz$$

(a) What is Length(γ_r)?
(b) Use the assumption of inequality 1 to show

$$|f(z)| \le Ar + B \qquad \text{on } \gamma_r$$

(c) Use part (b) to show

$$\left|\frac{f(z)}{z^{n+1}}\right| \le \frac{Ar+B}{r^{n+1}} \quad \text{on } \gamma_r$$

(d) Use part (b) and the Basic Integral Bound to show

$$|c_n| = \left|\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^{n+1}}\,dz\right| \le \frac{Ar+B}{r^n}$$

(e) Now take a limit as $r \to \infty$ to show $c_n = 0$ for $n \ge 2$. Why doesn't this work for $n = 1$? □

Anther of our major results is

Theorem 5. *Let U be a simply connected domain in $\mathbb{C}$ and let $f(z)$ be analytic in U. Then $f(z)$ has an antiderivative in U. That is there is a function $F(z)$ in U such that $F'(z) = f(z)$.* □

This was used to prove

Theorem 6. *Let U be a simply connected domain and let $f(z)$ be an analytic function on U that has no zeros in U then*

(a) $f(z)$ has an analytic logarithm in U. That is there is an analytic function $g(z)$ with $e^{g(z)} = f(z)$.

(b) For each nonzero integer n the function $f(z)$ has an analytic n-th root. That is there is an analytic function $h(z)$ with $h(z)^n = f(z)$. □

Problem 8. (15 points) It is important in the last two theorems that the domain be simply connected. So this by showing that when $U = \{z : 1 < |z| < 3\}$ that the function $f(z) = z$ has no analytic logarithm. *Hint:* If $e^{g(z)} = z$, then take a derivative to get $e^{g(z)}g'(z) = 1$. This implies $g'(z) = e^{-g(z)} = \dfrac{1}{z}$. Let γ be the circle $|z| = 2$. Then by the calculation we have just done

$$\int_\gamma g'(z)\,dz = \int_\gamma \frac{dz}{z} = 2\pi i.$$

(you may assume that $\int_\gamma \frac{dz}{z} = 2\pi i$ as we have done this calculation many times). Now explain why

$$\int_\gamma g'(z)\,dz = 0$$

which gives a contradiction. □

At this point I should give you some problems about radii of convergence of analytic functions. But we have done enough of those recently that I do not think we need to revisit the topic. Likewise with the mean value property of analytic and harmonic functions and the maximum modulus theorem.

Problem 9. (20 points) Use the Residue Theorem to find the following

(a)

$$\int_{|z-1|=1} \frac{e^z}{z^2-1}\,dz$$

(b)

$$\int_{|z|=42} \frac{e^z}{z^2-1}\,dz$$