Mathematics 552 Homework.

Some of these problems are review of what we have done in class. To start recall that we have derived the binomial theorem

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

where n is a positive integer

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

The numerator on this comes up often enough that is is convenient to have a special notation for it. For any complex number α and positive integer k the ***k-th falling power of*** α is

$$\alpha^{\underline{k}} = \overbrace{\alpha(\alpha-1)\cdots(\alpha-k+1)}^{k \text{ factors}}$$

and for $k=0$ we set

$$\alpha^{\underline{0}} = 1.$$

It is easy to see the pattern by looking at what happens for small k.

$$\begin{aligned}
\alpha^{\underline{0}} &= 1 \\
\alpha^{\underline{1}} &= \alpha \\
\alpha^{\underline{2}} &= \alpha(\alpha-1) \\
\alpha^{\underline{3}} &= \alpha(\alpha-1)(\alpha-2) \\
\alpha^{\underline{4}} &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)
\end{aligned}$$

Thus we can use this notation to write the binomial coefficients in the short hand notation

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}.$$

Here is anther pleasant use of this notation. Let $\alpha 0$ be a real number and let

$$f(x) = x^\alpha$$

and let use compute the first several derivatives of $f(x)$:

$$\begin{aligned}
f'(x) &= \alpha x^{\alpha-1} && = \alpha^{\underline{1}} x^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)x^{\alpha-2} && = \alpha^{\underline{2}} x^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)x^{\alpha-3} && = \alpha^{\underline{3}} x^{\alpha-3} \\
f^{(4)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)x^{\alpha-4} && = \alpha^{\underline{4}} x^{\alpha-4} \\
f^{(5)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)x^{\alpha-5} && = \alpha^{\underline{5}} x^{\alpha-5}
\end{aligned}$$

At this point we see the pattern.

Proposition 1. *Let $\alpha \neq$ be a real number and let*

$$f(x) = x^\alpha.$$

Then for any positive integer k the k-th derivative of $f(x)$ is

$$f^{(k)}(x) = \frac{d^k}{dx^k} x^\alpha = \alpha^{\underline{k}} x^{\alpha-k}.$$

Problem 1. Prove this. □

Recall that Taylor's Theorem is that if we have a series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

that converges around $x = 0$ then the coefficients c_k are given by the formula

$$c_k = \frac{f^{(k)}(0)}{k!}.$$

Let us apply this to the function

$$f(x) = (1+x)^\alpha.$$

By Proposition 1 (and the chain rule) the k-th derivative of this function is

$$f^{(k)}(x) = \alpha^{\underline{k}}(1+x)^{\alpha-k}.$$

(Exercise for you to test your understanding: how was the chain rule used?) Thus

$$f^{(k)}(0) = \alpha^{\underline{k}}(1+0)^{\alpha-k} = \alpha^{\underline{k}}.$$

Therefore if we have a convergent series

$$f(x) = (1+x)^\alpha = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

then

$$c_k = \frac{f^{(k)}(0)}{k!} = \frac{\alpha^{\underline{k}}}{k!} = \binom{\alpha}{k!},$$

where this defines the binomial coefficient $\binom{\alpha}{k!}$ for general α. Thus, at least formally, where have

Theorem 2 (Newton's Binomial Theorem for Fractional Exponents)**.** *Let α be a real number and x a number with $|x| < 1$. Then*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

□

Later this term we will show this converges (which you may already have done in your calculus class) and extend this to complex α and x.

Problem 2. Show that

$$\binom{\alpha}{0} = 1$$
$$\binom{\alpha}{1} = \alpha$$
$$\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$$
$$\binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{6}$$
$$\binom{\alpha}{4} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24}$$

and therefore the first terms of the series for $(1+x)^\alpha$ are

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + \cdots$$ □

Problem 3. Find the first four terms for the series (up to the x^3 term) for $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$. □

Problem 4. For $\alpha = -1$ show

$$\binom{-1}{k} = (-1)^k.$$

Hint: Here is what happens for $k = 5$, and you should be able to generalize to all k.

$$\begin{aligned}\binom{-1}{5} &= \frac{(-1)((-1)-1)((-1)-2)((-1)-3)((-1)-4)}{5!}\\ &= \frac{(-1)(-2)(-3)(-4)(-5)}{5!}\\ &= \frac{(-1)^5 5!}{5!}\\ &= (-1)^5.\end{aligned}$$ □

Using this problem in Newton's Binomial Theorem gives

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \cdots$$

As a check note this series is a geometric series and so we and work backwards:

$$1 - x + x^2 - x^3 + x^4 - \cdots = \frac{\text{first}}{1 - \text{ratio}} = \frac{1}{1-(-x)} = \frac{1}{1+x},$$

exactly what we started with, and which is reassuring that we are correct.

Here is some review of what we have done in class. We used Taylor's Theorem to derive the series expansions for real values x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

which we know from calculus converge for all real numbers x. We then used Euler's idea and extended the definitions of these functions to complex values by replanting the real variable x with a complex variable z, that is the definitions for complex values are

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \frac{z^8}{8!} \cdots$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots$$
$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \cdots$$

We will show later that these converge for all z. Then we used these series to prove

Theorem 3. *For any complex number z* ***Euler's Formula***

$$e^{iz} = \cos(z) + i\sin(z)$$

holds.

Problem 5. Prove this. *Hint:* I used the variable t in class, but other than changing the variable from t to z the same proof works. □

Problem 6. Use Euler's formula to show

(a) $e^{\pi i} = -1$.
(b) $e^{2\pi i} = 1$.
(c) $e^{\frac{\pi}{2} i} = i$.

Let θ be a real number. Then Euler's formula gives that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

The point $z = \cos(\theta) + i\sin(\theta)$ has x-coordinate $x = \cos(\theta)$ and y-coordinate $y = \sin(\theta)$. This is the points on the unit circle $|z| = 1$ which makes an angle of θ with the positive x-axis, just as in polar coordinates. See figure **??**

More generally if $z = x + iy$ is the point with polar coordinates r, θ, then we recall from vector calculus that

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

Thus

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + isin(\theta)) = re^{i\theta}.$$

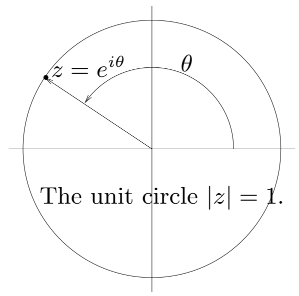

FIGURE 1. If $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$, then z is point on the unit circle defined by $|z| = 1$ with polar (as in polar coordinates) angle θ.

This is the ***polar form*** of the complex number z.

Problem 7. Find a polar form of the following complex numbers.

(a) -1,
(b) $-3i$,
(c) $1 - i$, and
(d) $-1 - i\sqrt{3}$.

One of the main results we done in the last couple of class meetings is that with the series definition of e^z the usual rule for exponents holds:

Theorem 4. *For any complex numbers z and w.*

$$e^{z+w} = e^z e^w.$$

Outline of proof. This is based on the Binomial Theorem in the form

$$(z + w)^n = \sum_{k+\ell=n} \frac{n!}{k!\ell!} z^k w^\ell$$

and a computation using the series for e^z and e^w

$$\begin{aligned}
e^z e^w &= \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{\ell=0}^{\infty} \frac{w^\ell}{\ell!}\right) \\
&= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots\right) \\
&= 1 + (z+w) + \left(\frac{z^2}{2!} + zw + \frac{w^2}{2!}\right) + \left(\frac{z^3}{3!} + \frac{z^2 w}{2!} + \frac{zw^2}{2!} + \frac{w^3}{3!}\right) \\
&\qquad + \left(\frac{z^4}{4!} + \frac{z^3 w}{3!} + \frac{z^2 w^2}{2!2!} + \frac{zw^3}{3!} + \frac{w^4}{4!}\right) + \cdots \\
&= 1 + \left(\sum_{k+\ell=1} \frac{z^k z^\ell}{k!\ell!}\right) + \left(\sum_{k+\ell=2} \frac{z^k z^\ell}{k!\ell!}\right) + \left(\sum_{k+\ell=3} \frac{z^k z^\ell}{k!\ell!}\right) \\
&\qquad + \left(\sum_{k+\ell=4} \frac{z^k z^\ell}{k!\ell!}\right) + \left(\sum_{k+\ell=5} \frac{z^k z^\ell}{k!\ell!}\right) + \cdots \\
&= \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} \frac{z^k w^\ell}{k!\ell!}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k+\ell=n} \frac{n!}{k!\ell!} z^k w^\ell\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\
&= e^{z+w}.
\end{aligned}$$

□

Proposition 5. *For any complex number z we have $e^z \neq 0$ and*

$$e^{-z} = \frac{1}{e^z}.$$

Proof. We know $e^0 = 1$ and using $e^{z+w} = e^z e^w$ with $w = -z$ gives

$$1 = e^{z-z} = e^z e^{-z}.$$

This implies $e^z \neq 0$ (for if it were 0 the product $e^z e^{-z}$ would be 0) and dividing by e^z gives $e^{-z} = \dfrac{1}{e^z}$. □

Proposition 6. *Let $z_1, z_2, \ldots, z_n$ be complex numbers. Then*

$$e^{z_1 + z_2 + \cdots z_n} = e^{z_1} e^{z_2} \cdots e^{z_n}.$$

Proof. We use induction. The base of the induction is $n = 1$, in which case the result reduces to $e^{z_1} = e^{z_1}$ and this is clearly true. As the induction hypothesis assume the result holds for some $n \geq 1$, that is

$$e^{z_1 + z_2 + \cdots z_n} = e^{z_1} e^{z_2} \cdots e^{z_n}.$$

and consider $e^{z+w} = e^z e^w$ with $z = z_1 + z_2 + \cdots z_n$ and $w = z_{n+1}$. Then

$$\begin{aligned} e^{z_1+z_2+\cdots z_n+z_{n+1}} &= e^z e^w \\ &= e^z e^w \\ &= e^{z_1+z_2+\cdots z_n} e^{z_{n+1}} \\ &= e^{z_1} e^{z_2} \cdots e^{z_n} e^{z_{n+1}} \qquad \text{(by induction hypothesis)} \end{aligned}$$

but this shows the results for $n+1$, which closes the induction and completes the proof. □

Proposition 7. *Let n be an integer (either positive or negative), then for any $z \in \mathbb{C}$*

$$e^{nz} = (e^z)^n \,.$$

Proof. For $n = 0$ this reduces to $1 = 1$ which is true. If $n > 0$ then by Proposition 6 with $z_1 = z_2 = \cdots = z_n$ we have

$$e^{nz} = e^{\overbrace{z + z + \cdots + z}^{n \text{ terms}}} = \overbrace{e^z e^z \cdots e^z}^{n \text{ factors}} = (e^z)^n \,.$$

For $n < 0$ we use that $-n > 0$ and use Proposition 5

$$e^{nz} = e^{-(-nz)} = \frac{1}{e^{-nz}} = \frac{1}{(e^z)^{-n}} = (e^z)^n$$

□

Problem 8. Use that $e^{3i\theta} = \left(e^{i\theta}\right)^3$ to find formulas for

(a) $\cos(3\theta)$

(b) $\sin(3\theta)$.