Mathematics 552 Homework.

We have shown Euler's formula

$$e^{iz} = \cos(z) + i\sin(z)$$

If we replace z by $-z$ and use that $\cos(-z) = \cos(z)$ (that is cos is an even function) and $\sin(-z) = -\sin(z)$ (thus $\sin(z)$ is an odd function) we get

$$e^{-z} = \cos(z) - i\sin(z).$$

Problem 1. Use these equation to show

$$\cos(z) = \frac{e^{-z} + e^{--z}}{2}$$

$$\sin(z) = \frac{e^{-iz} - e^{-iz}}{2i} \qquad \square$$

Problem 2. Use the formulas of the previous problem to show that

$$\cos^2 z + \sin^2 z = 1. \qquad \square$$

Problem 3. Let $a = -6 + 6i$.

(a) Write a in polar form $a = re^{i\theta}$.

(b) Use the polar form to find a^9 and write the result in the form iy.

(c) Use the polar form to fin all cube roots of a, where are three, and write the results in the form $x + iy$. $\square$

Let U be an open subset of the complex plane U and let $f\colon U \to \mathbb{C}$ be a complex valued function defined on U. Then, in analogy with the definition in calculus, it is natural to define f to be ***differentiable*** at $z \in U$ if and only if the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

exits.

It is easy to check that many of the functions are are familiar with, such as polynomials, are differentiable at all points. Here is an example

Problem 4. Let $f(z) = z^4$. Use the Binomial Theorem to expand $(z + \Delta z)^4$ in the difference quotient

$$\frac{f(z + \Delta z) - f(z)}{\Delta z}$$

simplify the result and cancel the Δz out of the denominator to show that $f'(z) = 4z^3$. $\square$

Problem 5. For a function that takes a bit more work let

$$f(z) = \frac{1}{z^2}$$

and show directly from the limit definition of $f'(z)$ that

$$f'(z) = \frac{-2}{z^3}$$

at all points where $z \neq 0$. *Hint:* One way to start is to compute

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{\Delta z}\left(\frac{1}{(z+\Delta z)^2} - \frac{1}{z^2}\right) = \frac{1}{\Delta z}\left(\frac{z^2-(z+\Delta z)^2}{z^2(z+\Delta z)^2}\right)$$

and expand the numerator. You should then be able to cancel out the Δz in the denominator so that taking the limit as $\Delta z \to 0$ becomes easy. □

Problem 6. Now let $f(z)$ be the function

$$f(z) = e^{az}$$

where a is a complex constant. Then

$$f(z) = 1 + (az) + \frac{(az)^2}{2!} + \frac{(az)^3}{3!} + \frac{(az)^4}{4!} + \frac{(az)^5}{5!} + \cdots$$

In this problem you will compute the derivative of $f(z)$ at $z = 0$. Verify that the difference quotient for f at $z = 0$ can be simplified as follows:

$$\begin{aligned}
\frac{f(0+\Delta z)-f(0)}{\Delta z} &= \frac{f(\Delta z)-1}{\Delta z} \\
&= \frac{1}{\Delta z}\left(e^{a\Delta z}-1\right) \\
&= \frac{1}{\Delta z}\left(\left(1+(a\Delta z)+\frac{(a\Delta z)^2}{2!}+\frac{(a\Delta z)^3}{3!}+\frac{(a\Delta z)^4}{4!}+\cdots\right)-1\right) \\
&= \frac{1}{\Delta z}\left((a\Delta z)+\frac{(a\Delta z)^2}{2!}+\frac{(a\Delta z)^3}{3!}+\frac{(a\Delta z)^4}{4!}+\cdots\right) \\
&= a + \frac{a^2\Delta z}{2!} + \frac{a^3(\Delta z)^2}{3!} + \frac{a^4(\Delta z)^3}{4!} + \cdots
\end{aligned}$$

And use this to show that

$$\begin{aligned}
f'(0) &= \lim_{\Delta z\to} \frac{f(0+\Delta z)-f(0)}{\Delta z} \\
&= \lim_{\Delta z\to 0}\left(a + \frac{a^2\Delta z}{2!} + \frac{a^3(\Delta z)^2}{3!} + \frac{a^4(\Delta z)^3}{4!} + \cdots\right) \\
&= a + 0 + 0 + 0 + \cdots \\
&= a.
\end{aligned}$$

□

The main point of the previous problem can be summarized as

$$\lim_{\Delta z\to 0}\frac{e^{a\Delta z}-1}{\Delta z} = a. \tag{1}$$

Problem 7. Let $f(z) = e^{az}$. Use the Limit (1) to show that

$$f'(z) = ae^{az}.$$

Hint: We know that the exponential function satisfies $e^{z_1+z_2} = e^{z_1}e^{z_2}$. Use this to show

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \left(\frac{e^{a\Delta z}-1}{\Delta z}\right)e^{az}.$$

and hopefully it is clear what to do from here. $\square$

So we now have that for any complex constant that

$$\frac{d}{dz}e^{az} = ae^{az}.$$

Problem 8. Use this and

$$\sin(az) = \frac{e^{iaz} - e^{-iaz}}{2i}, \qquad \cos(az) = \frac{e^{iaz} + e^{-iaz}}{2}$$

to find formulas for the derivatives of $\sin(az)$ and $\cos(az)$. $\square$

Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ be defined in an open set U where u and v are the real and imaginary parts of f. Recall from vector calculus that the partial derivatives of u and v are defined by

$$\begin{aligned}
\frac{\partial u}{\partial x}(x, y) &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \\
\frac{\partial u}{\partial y}(x, y) &= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \\
\frac{\partial v}{\partial x}(x, y) &= \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\
\frac{\partial v}{\partial x}(x, y) &= \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta z}
\end{aligned}$$

Problem 9. Letting $f(z) = u(x, y)+iv(x, y)$ as above and letting $z = x+iy$ and $\Delta z = \Delta x + i\Delta y$. Show that $(f(z + \Delta z) - f(z))/\Delta z$ can be written as

$$\begin{aligned}
\frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} \\
&\qquad + i\,\frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y}
\end{aligned}$$

$\square$

Problem 10. Let $f(z)$ be as in the previous problem and assume that f is differentiable at some point of U. We now compute

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

in two ways.

(a) First we let $\Delta z \to 0$ through real values, that is let $\Delta y = 0$ so that $\Delta z = \Delta x$ and compute $f'(z)$ by the formula

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x}$$

and use the formula of Problem 9 to show

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\,\frac{\partial v}{\partial x}(x, y).$$

(b) Now take the limit as $\Delta z \to 0$ through imaginary values. That is we let $\Delta x = 0$ so that $\Delta z = i\Delta y$. Then

$$f'(z) = \lim_{\Delta y \to 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y}.$$

Now use the formula of Problem 9 to show using this limit that we also have the formula

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y)$$

(c) If f is differentiable at a point, these two formulas for the derivative must be equal. Use this to show

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

□

You have just proven one of the most basic and important results in complex analysis:

Theorem 1. *If $f(z) = u + iv$ is defined on an open subset of $\mathbb{C}$, then at any point where the complex derivative $f'(z)$ exists the* ***Cauchy-Riemann equations***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold. □

Therefore at any point where the Cauchy-Riemann equations do not hold the function is not complex differentiable.

Problem 11*.* Show that $f(z) = (x + 2y) + i(-3x + 7y)$ is not differentiable at any points. *Hint:* In this case $u = x + 2y$ and $v = -3x + 7y$. Show that the Cauchy-Riemann equations do not hold at any point. □