Mathematics 552 Homework.

We have defined a function $f(z)$ defined on an open subset U of $\mathbb{C}$ to be ***analytic*** if and only if its complex derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists at all points of U. We have also outlined proofs that the basic derivative formulas from differential calculus hold. That is if f and g are analytic in U and c_1 and c_2 are constants the sum

$$h(z) = c_1 f(x) + c_2 g(z)$$

is analytic with the expected derivative

$$h'(z) = c_1 f'(z) + c_2 g'(z).$$

Likewise the product

$$h(z) = f(z)g(z)$$

is analytic and the product rule

$$\frac{d}{dz}\left(f(z)g(z)\right) = f'(z)g(z) + f(z)g'(z)$$

holds. Also if

$$h(z) = \frac{f(z)}{g(z)}$$

then $h(z)$ is analytic on the set of points of U where $g(z) \neq 0$ and the quotient rule

$$h'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

holds. Using these rules we see that any polynomial

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$$

is analytic on all of $\mathbb{C}$ with the usual formula for the derivative holding. Recall a rational function is one of the form

$$f(z) = \frac{p(z)}{g(z)}$$

where $p(z)$ and $q(z)$ are polynomials. This is analytic at all points of $\mathbb{C}$ where $q(z) \neq 0$.

We have also seen that for any complex constant a that

$$\frac{d}{dz} e^{az} = a e^{az}.$$

Using this and that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{1}$$

we found the usual derivative formulas for these functions hold:

$$\frac{d}{dz}\cos(z) = -\sin(z)$$
$$\frac{d}{dz}\sin(z) = \cos(z).$$

Problem 1. Use the formulas (1) to show that

$$\cos^2(z) + \sin^2(z) = 1$$

holds for all complex numbers z.

Problem 2. (a) Use that $e^{2\pi i} = 1$ and the formulas above for $\cos(z)$ and $\sin(z)$ to show

$$\cos(z + 2\pi) = \cos(z), \qquad \sin(z + 2\pi) = \sin(z).$$

That is both cos and sin are periodic with period 2π. (Note this is more general than the version you learned in high school as it holds for all complex z and not just real numbers.)

(b) Use that $e^{\pi i} = -1$ to show

$$\cos(z + \pi) = -\cos(z), \qquad \sin(z + \pi) = -\sin(z).$$

More generally we can use the formulas (1) for cos and sin and that $e^{z+w} = e^z e^w$ to prove the complex forms of the addition formulas that you all know:

$$\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$
$$\sin(z + w) = \cos(z)\sin(w) + \sin(z)\cos(w).$$

If you do not remember these formulas you should do the calculation showing that they holds as a way to remember them for the rest of this term.

We make the usual definition of $\tan(z)$ and $\cot(z)$,

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \qquad \cot(z) = \frac{\cos(z)}{\sin(z)}.$$

Problem 3. (a) Use the derivative formulas for sin and cos and the quotient rule to show

$$\frac{d}{dz}\tan(z) = \frac{1}{\cos^2(z)}, \qquad \frac{d}{dz}\cot(z) = \frac{-1}{\sin^2(z)}.$$

Here are a couple of other functions that will be useful to us.

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$
$$\sinh(z) = \frac{e^z - e^{-z}}{2}.$$

Problem 4. Prove the following:

(a)

$$\cosh^2(z) - \sinh^2(z) = 1.$$

(b)

$$\frac{d}{dz}\cosh(z) = \sinh(z), \qquad \frac{d}{dz}\sinh(z) = \cosh(z).$$

(c)

$$\cos(iz) = \cosh(z).$$

(d)

$$\sin(iz) = i\sinh(z).$$

Let $z = z + iy$. Then using the addition formulas for cos and sin we have

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) \tag{2}$$

$$\sin(x+iy) = \sin(x)\cos(iy) + \sin(iy)\cos(x). \tag{3}$$

Problem 5. (a) Let

$$\cos(x+iy) = u(x,y) + iv(x,y)$$

with $u(x,y)$ and $v(x,y)$ real valued. Use the formula (2) and Problem 4 to give formulas for $u(x,y)$ and $v(x,y)$.

(b) As a check on the answer show directly that your u and v satisfy the Cauchy-Riemann equations.

Problem 6. (a) Let

$$\sin(x+iy) = u(x,y) + iv(x,y)$$

with $u(x,y)$ and $v(x,y)$ real valued. Use the formula (3) and Problem 4 to give formulas for $u(x,y)$ and $v(x,y)$.

(b) As a check on the answer show directly that your u and v satisfy the Cauchy-Riemann equations.

We have defined some new functions. If $z \neq 0$, then we have defined the ***complex logarithm*** as

$$\log(z) = \ln(|z|) + i\arg(z)$$

where ln is the natural (that is base e) logarithm from calculus. Because arg is not single valued the function log is not single valued. For example if $z = re^{i\theta}$, then

$$\log(z) = \ln(r) + i\theta + 2\pi ni.$$

where n varies over the integers. To summarize some of the calculations we did in class:

Proposition 1. *If $z = re^{i\theta}$ with $r > 0$ then*

$$\log(z) = \ln(r) + i\theta + 2n\pi i$$

with $z \in \mathbb{Z}$ gives all solutions for w in

$$e^w = z.$$

□

Proposition 2. *For all $z \neq 0$ we have*

$$e^{\log(z)} = z$$
$$\log(e^z) = z + 2n\pi i.$$ □

Problem 7. Find $\log(z)$ for the following values of z.
(a) $z = -3i$
(b) $z = -5 - 5i$
(c) $z = \sqrt{3} - i$. □

Consider a complex number $z = x + iy$, with $x > 0$. Then when we write $z = re^{i\theta}$ we can choose the angle θ with $-\pi/2 < \theta < \pi/2$. See Figure 1.

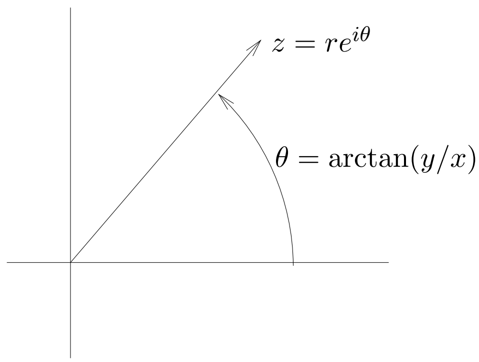

Figure 1. For complex numbers z with $x > 0$ we can use for the polar angle $\theta = \arctan(y/x)$ chosen with $-\pi/2 < \theta < \pi/2$.

Then we can work with the special case of $\log(z)$ with $n = 0$, that is let us temporarily use the definition for z with $\mathrm{Re}(z) > 0$

$$\begin{aligned}\log(z) &= \ln|z| + i\arg(z)\\ &= \ln(\sqrt{x^2+y^2}) + i\arctan(y/x)\\ &= \frac{1}{2}\ln(x^2+y^2) + i\arctan(y/x).\end{aligned}$$

Problem 8. For this function

$$f(z) = \log(z) = u + iv = \frac{1}{2}\ln(x^2+y^2) + i\arctan(y/x)$$

show that the Cauchy-Riemann equations hold. □

Problem 9. Let $a > 1$ be a real number.
(a) Find all complex numbers z with

$$\cos(z) = a.$$

(b) Find all complex numbers z with

$$\sin(z) = a.$$ □