Mathematics 552 Homework.

Judging from the test, one of the topics we needs some review is arg vs Arg and log vs Log. Let z be a complex number, $z \neq 0$. Then it can be written in polar form

$$z = re^{i\theta}$$

where $r > 0$ and θ is a real number. The number r is uniquely determined by z and in fact

$$r = |z|$$

is just the length (which we also call the modulus or absolute value) of z. But the number θ is not uniquely determined by z. If $n \in Z$ (that is an integer, positive, negative, or zero), the $e^{2n\pi} = 1$ and so

$$re^{i\theta + 2n\pi i} = re^{i\theta}e^{2n\pi i} = z \cdot 1 = z.$$

Therefore all of the values $\theta + 2n\pi$ work for the angle in the polar form of $z = re^{i\theta}$. We call the angle θ an ***argument*** of z. And to remind ourselves that there are many choices we write

$$\arg(z) = \theta + 2n\pi$$

where the $= 2n\pi$ is rather like using the $+C$ when doing an integral, it reminds you that there are many choices.

In is often useful to have have a unique choice of the angle. So the ***principle value of the argument*** is the choice of the angle θ with

$$-\pi < \theta \leq \pi$$

and is denoted as

$$\operatorname{Arg}(z) = \theta \qquad \text{where} \qquad r = |z|, \quad \text{and} \quad -\pi < \theta \leq \pi$$

Thus

$$\arg(-1 - i) = \frac{5\pi}{4} + 2n\pi = \frac{-2\pi}{4} + 2n\pi$$

and

$$\operatorname{Arg}(-1 - i) = \frac{-2\pi}{4}.$$

Let ln be the logarithm from calculus. That is it is defined on positive real numbers t and

$$\ln(e^x) = x$$

for all real numbers x and

$$e^{\ln t} = t$$

for all positive real numbers. Let z have the polar form

$$z = re^{i\theta} = |z|e^{i\theta}.$$

Set

$$w = \ln(r) + i\theta$$

then, to repeat a calculation we have done before,

$$e^w = e^{\ln(r) + i\theta} = e^{\ln(r)}e^{i\theta} = re^{i\theta} = z.$$

Thus if we define the complex logarithm as

$$\log(z) = \ln(|z|) + i\arg(z)$$

that is

$$\log(z) = \ln(|z|) + i\theta + 2n\pi i$$

Then, like arg, the complex logarithm is not unique, we we have to add in the $2n\pi i$. As we have said before this makes log multivalued.

The calculation we have just done shows

$$e^{\log(z)} = z.$$

We also have

$$\log(e^z) = z + 2n\pi i.$$

And like the for arg it is often nice to have make a choice of just one of the values of $\log(z)$. This the ***principle value of the logarithm*** of z is

$$\operatorname{Log}(z) = \ln(|z|) + i\operatorname{Arg}(z).$$

Problem 1. Compute the following

(a) $\arg(-3+3i)$,
(b) $\operatorname{Arg}(-3+3)$,
(c) $\log(-3+3i)$, and
(d) $\operatorname{Log}(-3+3i)$. □

Problem 2. If z is nonzero complex number and α is an complex number our offical definition of z^α is

$$z^\alpha = e^{\alpha\log(z)}.$$

In general this is multivalued because log is multivalued. If we want just one value, then one solution is to use the ***principle value*** of z^α which is

$$z^{\alpha\operatorname{Log}(z)}.$$

Compute the principle values of the following

(a) 1^α,
(b) 2^i,
(c) i^i, and
(d) i^{1+1}. □

Here are a few more problems reviewing some earlier topics.

Problem 3. (a) Draw a picture of the region

$$D = \{z : 1 \le |z| \le 2, 0 \le Arg z \le \pi/2\}.$$

(b) Let $f(z) = z^2$. Draw the image of D under the function f. That is draw

$$f[D] = \{f(z) : z \in D\}.$$

(c) Let $g(z) = \dfrac{1}{z}$. Draw the image of D under $g(z)$.
(d) Let $h(z) = z^4$. Draw the image of D under $h(z)$. □

Problem 4. (a) Draw the set

$$S = \{z : 0 < \mathrm{Re}(z) < 5\}.$$

(b) Draw the image of S under the function $f(z) = e^z$. □