Mathematics 552 Homework.

Here are some power series we know:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} \cdots \tag{1}$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots \tag{2}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots \tag{3}$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - \cdots \tag{4}$$

We can combine these with some easy tricks to get the series for some more complicated functions. For example to get the series $\sin(z^3)$ we replace z by z^3 in equation (2) to get

$$\begin{aligned}
\sin(z^3) &= \sum_{n=0}^{\infty} \frac{(-1)^n (z^3)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{6n+3}}{(2n+1)!} \\
&= z^3 - \frac{z^9}{3!} + \frac{z^{15}}{5!} - \frac{z^{21}}{7!} + \frac{z^{27}}{9!} - \cdots
\end{aligned}$$

Then if we wanted the series for the function

$$F(z) = \int_0^z \sin(t^3)\, dt$$

we can just integrate term at a time to get

$$\begin{aligned}
F(z) &= \int_0^z \sin(t^3)\, dt \\
&= \int_0^z \left(t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} - \frac{t^{21}}{7!} + \frac{t^{27}}{9!} - \cdots \right) dt \\
&= \frac{z^4}{4} - \frac{z^{10}}{3!(10)} + \frac{z^{16}}{5!(16)} - \frac{z^{22}}{7!(22)} + \frac{z^{28}}{9!(28)} - \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{6n+4}}{(2n+1)!(6n+4)}.
\end{aligned}$$

Problem 1. Find the power series for the following functions:
(a) e^{-2z},
(b) $\cos(3z^2)$,
(c) $\log(1+z) - \displaystyle\int_0^z \frac{dt}{1+z}$.

(d) $\displaystyle\int_0^z e^{3t^2}\,dt$.

In class we used these methods to show

$$\begin{aligned}\arctan(z) &= \int_0^z \frac{dt}{1+t^2}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}\\ &= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} - \cdots\end{aligned}$$

This has radius of convergence $R = 1$.

To compute π we can use the series for the arctan For this to be efficient we wish to use values of z that are close to zero. To get a reasonably rapidly convergent series note if

$$\alpha = \arctan(1/2), \qquad \beta = \arctan(1/3)$$

then using the addition angle for the tangent we havd

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \frac{1/2+1/3}{1-(1/2)(1/3)} = 1.$$

Therefore

$$\alpha + \beta = \frac{\pi}{4}$$

which implies

$$\pi = 4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right) = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}}\right).$$

Stopping this series at $k = 13$ gives the value of π to 10 decimal places.

In 1796 John Machin showed that[1]

$$\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239},$$

which leads to the series

$$\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{16}{5^{2k+1}} - \frac{4}{239^{2k+1}}\right)$$

which converges much faster. Using 73 terms gives π to 100 hundred decimal places.

Using the following variant on this theme, that

$$\pi = 48\arctan\frac{1}{49} + 128\arctan\frac{1}{57} - 20\arctan\frac{1}{239} + 48\arctan\frac{1}{110443},$$

was used by Yasumasa Kanada of Tokyo University in 2002 to compute π to 1,241,100,000,000 digits.

[1] If you wish to prove this, probably the easiest way is to notice that $(5+i)^4(239-i) = 114244(1+i)$ and use the polar form of complex numbers to get the result.

For a modern method there is the formula found in 1995 by Simon Plouffe:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

which nice form the point of view of computing as powers of 16 are very easy to compute in hexadecimal. In particular using the first n terms of this series gives at least the first n-hexadecimal digits of π. $\square$

Problem 2. Find the radius of convergence of following points about the indicated points.

(a) The function $f(z) = \dfrac{e^{2z+4}}{z^2+9}$ about the point $z = 4$.

(b) The function $f(z) = \dfrac{\cos(z)}{z^2+2z+2}$ about $z = 3+4i$.

(c) The function $f(z) = \dfrac{\sin(z)}{e^z-1}$ about $z = 1+2i$.

Theorem 1 (Mean Value Property of Analytic Functions)**.** *Let $f(z)$ be analytic in the domain D, and assume that the closed disk with center a and radius r (that is $\{z : |z-a| \le r\}$) is contained in D. Then*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt. \tag{5}$$

(This can be loosely restated as: The average of an analytic function on a circle is equal to the value of the function at the center of the circle.)

Problem 3. Prove this by doing the following.

(a) Show (and this is not much more that a restatement of the Cauchy Integral Formula for a disk)

$$f(a) = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{z-a}\,dz.$$

(b) In this integral parameterize the circle $|z-a| = r$ by $z = a + re^it$ and do the substitution. This should lead to the desired result. $\square$

Remark: At one time it was standard terminology to refer to the "average value" as the "mean value" (and still is in statistics and probability). This is where the term "mean value property" comes from.

Problem 4. The mean value property is sometime written as follows. With the same hypothesis as in Theorem 1

$$f(a) = \frac{1}{2\pi r}\int_{|z-a|=r} f(z)\,ds \tag{6}$$

where ds is arc length along the circle $|z-a| = r$. (Note that $2\pi r$ is the length of the circle $|z-a| = r$ so that this is still expressing $f(a)$ as the average value of $f(z)$ over the circle $|z-a| = r$.) Show that this follows from Theorem 1 by a change of variable as follows

(a) Let $z = a + re^{it}$. Then we know that $ds = |dz| = |z'(t)|\,dt$. What is $|z'(t)|\,dt$ in this case?
(b) To the change of variable $z = a + re^{it}$ in equation (5) to deduce equation (6) holds. $\square$

First some review.

Proposition 2. *Let $f = u + iv$ be analytic in a connected domain D. Assume that $|f(z)|$ is constant. Then $f(z)$ is constant.*

Problem 5. Prove this along the following lines.
(a) If $|f(z)|$ is constant then show $u^2 + v^2 = c$ for some real constant c.
(b) If $c = 0$ show $f(z)$ is the constant function 0.
(c) If $c \neq 0$ use the Cauchy-Riemann equations to show $f(z)$ is constant.

We have outlined a proof of

Proposition 3. *Let $f(z)$ be continuous on the circle $|z - z_0| = r$. Then*

$$\left|\int_0^{2\pi} f(z_0 + re^{it})\,dt\right| \leq \int_0^{2\pi} |f(z_0 + re^{it})|\,dt$$

and if equality holds, then $|f(z_0 + re^{it})|$ is constant (as a function of t.) $\square$

In the following $D(z_0, r)$ is the disk

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Theorem 4 (Maximum modulus principle)**.** *Let $f(z)$ be analytic on the closure of $D(z_0, R)$ and $\partial D(z_0, R)$ and assume $|f(z)|$ has a maximum at $z = z_0$ (that is $|f(z)| \leq |f(z_0)|$ for $z \in D(z_0, R)$). Then $f(z)$ is constant in $D(z_0, R)$.*

Problem 6. Prove this along the following lines.
(a) If $0 < r < R$ use the mean value property of analytic functions to write

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt.$$

(You don't have to prove this). Then use the argument we gave in class to show

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\,dt \leq |f(z_0)|.$$

(b) Explain why for equality to hold in the second of these inequalities we have $|f(z_0 + re^{it})| = |f(z_0)|$ for $0 \leq t \leq 2\pi$.
(c) By varying $r \in (0, R)$ and $t \in [0, 2\pi]$ in part (b) show that $|f(z)| = |f(z_0)|$ for $z \in D(z_0, R)$.
(d) Now use Proposition 2 to show $f(z)$ is constant in $D(z_0, R)$.

Here is anther form of the maximum modulus principle.

Problem 7. Let D be a bounded domain and let $f(z)$ be analytic on D and ∂D Then $f(z)$ achieves its maximum on $D \cup \partial D$ is on the boundary, ∂D.

Problem 8. Prove this along the following lines.

(a) If $|f(z)|$ is constant, then $f(z)$ is constant and so the maximum of $|f(z)|$ occurs at all points of ∂D. In particular it occurs on the boundary.

(b) So assume that $f(z)$ is not constant. Assume, toward a contradiction, that the maximum of $|f(z_0)|$ occurs in D rather than on ∂D. Then get a contradiction by showing that $f(z)$ is constant.

Proposition 5 (Minimum modulus principle)**.** *Let $f(z)$ be analytic on the closure of $D(z_0, R)$ and assume that $|f(z)|$ has a minimum at $z = z_0$ (that is $|f(z)| \geq |f(z_0)|$ for $z \in D(z_0, R)$). Then either $f(z)$ is constant or $f(z_0) = 0$.*

Problem 9. Prove this. *Hint:* If $f(z_0) \neq 0$ then show $f(z) \neq 0$ for all $z \in D(z_0, R)$ and then apply the maximum modulus principle to $g(z) = 1/f(z)$.