Math 552 Test 2.

- *This is due on Tuesday, March 23 by midnight. It should be submitted via Blackboard as a pdf document.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *Please print your name on the first page of the test.*
- *Since you have plenty of time on this test you should submit neat papers. By this I do not mean handwriting, but more not having crossed out work and also taking the time to write sentences explaining what you are doing. If you are writing the paper by hand, it is good idea to make a rough draft to get the details correct before making the final copy.*
- *Also solutions with all formulas and no English explaining what is going on will not receive full credit.*

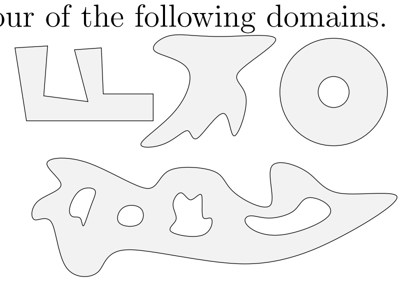
We have proven:

Theorem 1 (Cauchy Integral Theorem)**.** *Let D be a bounded domain in $\mathbb{C}$ with nice boundary ∂D, which we transverse in the direction so that the interior of D is on the left. Let $f(z)$ be a function which is analytic in D and on the boundary ∂D. Then*

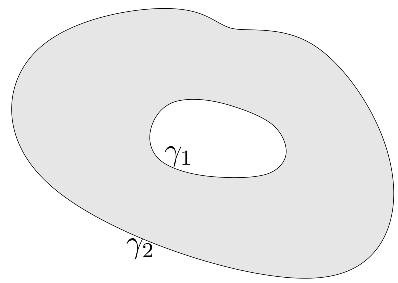
$$\int_{\partial D} f(z)\,dz = 0.$$

□

Note that this hold even when D is not simply connected. For example it holds for four of the following domains.



Problem 1. (10 points) Let γ_1 and γ be two simple closed curves with γ_1 inside of γ_2 and D the domain which is between the two curves as shown.



(a) Sketch this domain on your exam and put arrows on γ_1 and γ_2 showing which way they are being traversed.
(b) Let $f(z)$ be a function that is analytic on γ_1, γ_2, and D. Use Cauchy's Integral Theorem to show

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Note that this is being graded in part for how it is written. So formulas with no English will be graded down. You should say why Cauchy's Theorem applies and why the integrals have have the signs that they do. □

Two more of our Theorems are

Theorem 2. *Let $f(z) = u + iv$ be defined in an open set U and such that the partial derivatives u_x, u_y, v_x, and v_y are continuous and satisfy the Cauchy-Riemann equations*

$$u_x = v_y$$
$$u_y = -v_x$$

Then $f(z)$ is analytic in U. □

Theorem 3. *Let $f(z)$ be analytic in simply connected open set U. Then there is a function $F(z)$ with $F'(z) = f(z)$ in U. (That is an analytic function in a simply connected domain has an antiderivative in that domain.)* □

Recall that a real valued function h defined on an open subset U of $\mathbb{C}$ is ***harmonic*** if and only if

$$h_{xx} + h_{yy} = 0.$$

On Test 1 you proved the following

Proposition 4. *If h is harmonic on the open set U and we set*

$$u = h_x, \qquad v = -h_y$$

then $f(z) = u + iv$ satisfies the Cauchy-Riemann equations. $\square$

We will now prove

Theorem 5. *Let h be harmonic in a simply connected open set U. Then there is an analytic function $H(z)$ with*

$$\mathrm{Re}(H(z)) = h(z)$$

in U. (That is in a simply connected domain every harmonic function is the real part of an analytic function.)

Problem 2. (20 points) Prove this along the following lines

(a) Let

$$f(z) = h_x - ih_y$$

and explain why Proposition 4 together with Theorem 2 imply $f(z)$ is analytic in U.

(b) Explain why $f(z)$ has an antiderivative in U. Call this antiderivative $F(z)$. That is $F'(z) = f(z)$ and write

$$F(z) = U + iV.$$

(c) Use that the derivative of $F(z)$ can be computed by the formula

$$F'(z) = U_x + iV_x$$

and the Cauchy-Riemann equations to show

$$U_x = h_x, \qquad U_y = h_y$$

(d) Explain why $U(z) = h(z) + c$ where c is constant and use this to finish the proof. $\square$

Anther of our recent results is

Theorem 6. *Let $f(z)$ be analytic in the simply connected open set U and assume $f(z) \neq 0$ for all z in U. Then $f(z)$ has an analytic logarithm in U. That is there is an analytic function $g(z)$ in U such that $f(z) = e^{g(z)}$.* $\square$

Also recall that we have defined $\cosh(z)$ and $\sinh(z)$ by

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \frac{e^z - e^{-z}}{2}.$$

Problem 3. (20 points) Let $f(z)$ and $g(x)$ be analytic functions in on a simply connected open set U such that

$$f(z)^2 - g(z)^2 = 1$$

for all $z \in U$. Show that there is an analytic function $h(x)$ such that

$$f(z) = \cosh(h(z)), \qquad g(z) = \sinh(h(z)).$$

Hint: To start factor and rewrite the equation as

$$(f(z) + g(z))(f(z) - g(z)) = 1.$$

(a) Explain why the functions $f(z)+g(z)$ and $f(z)-g(z)$ never vanish on the set U.

(b) Use part (a) to explain why there are analytic functions $h_1(z)$ and $h_2(z)$ defined on U with

$$f(z) + g(z) = e^{h_1(z)}$$
$$f(z) - g(z) = e^{h_2(z)}.$$

(c) Show that $e^{h_1(z)+h_2(z)} = 1$ and therefore $h_1(z) + h_2(z) = 2n\pi i$ for some integer n.

(d) Show that if $h(z) = h_1(z)$ then

$$f(z) + g(z) = e^{h(z)}$$
$$f(z) - g(z) = e^{-h(z)}.$$

(e) Now finish the proof. □

Problem 4. (10 points) Find the radius of convergence of the following series:

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k z^k}{k+5}$.

(b) $\displaystyle\sum_{n=0}^{\infty} n^3 3^n (z+1+2i)^{2n}$.

Using that

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} = 1 + w + w^2 + w^3 + w^4 + \cdots \tag{1}$$

(which converges for $|w| < 1$) we have found power series expansions of some rational functions. For example let us find the expansion of

$$\frac{1}{5-z}$$

about $z = 3$, that is in powers of $(z-3)$.

$$\begin{aligned}
\frac{1}{5-z} &= \frac{1}{5-(z-3+3)} \\
&= \frac{1}{2-(z-3)} \\
&= \frac{1}{2}\left(\frac{1}{1-\left(\frac{z-3}{2}\right)}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z-3}{2}\right)^k \qquad\qquad \text{Using (1) with } w = \frac{z-3}{2} \\
&= \sum_{k=0}^{\infty}\frac{(z-3)^k}{2^{k+1}}.
\end{aligned}$$

This converges when

$$|w| = \left|\frac{z-3}{2}\right| < 1$$

that is when $|z-3| < 2$. Therefore the radius of convergence is 2.

Problem 5. (10 points) Find the power series expansion of

$$f(z) = \frac{1}{10-z}$$

about the points $z = 1 + 2i$ and give its radius of convergence. □

Problem 6. (10 points) Find the power series expansion of

$$\frac{1}{w-z}$$

about the complex number $z = c \neq w$ and give its radius of convergence. □

Anther big result we have done since the last test is

Theorem 7 (Liouville's theorem)**.** *A bounded entire function is constant.* □

Problem 7. (10 points) Let $f(z)$ be an entire function with $|f(z)| \geq 1$ for all z. Use Liouville's Theorem to show $f(z)$ is constant. □

Problem 8. (10 points) Let $f(z)$ and $g(z)$ be two entire functions such that

$$|f(z)| \geq 1 + |g(z)|$$

for all $z \in \mathbb{C}$. Show $f(z)$ and $g(z)$ are both constant. □