Math 552 Test 3.

- *This is due on Tuesday, April 20 by midnight. It should be submitted via Blackboard as a pdf document and should have your name on the first page.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *You will be graded in part on writing problems up correctly. In particular you can lose points with answers that are all formulas and equations without any English.*

Some remarks about ways to lose points.

- Inequalities only make sense between real numbers. Thus for complex numbers z and w the inequality $|z| < |w|$ makes sense (as $|z|$ and $|w|$ are real numbers) but $z < w$ does not make sense. Putting inequality symbols between complex numbers will cost you points.
- Related to the previous point, the radius of a circle is always a positive number. So saying that the radius of convergence of a series is $R = 4 + 5i$ is obviously wrong and what is probably met is $R = |4 + 5i| = \sqrt{4^2 + 5^2} = \sqrt{41}$.
- Bad algebra such as

$$\sqrt{a^2 + b^2} = a + b$$
$$e^{2z} = 2e^z$$
$$\frac{\sin 4z}{4} = \sin(z).$$

  This type of bad algebra will definitely lose points.

Problem 1. (10 points)

(a) Find the radius of convergence of the Taylor expansion of

$$f(z) = \frac{\sin(z^3 + 42)}{z^2 - 2z + 5}$$

about the point $z_0 = -3 + 6i$.

(b) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n^2+1)(z-1+2i)^{3n+1}}{5(27)^n}$$

□

Many of the following problems are based on

Theorem 1 (The Residue Theorem)**.** *Let $f(z)$ be analytic on a simple closed curve γ and also analytic inside of γ except at a finite number of isolated singularities $z_1, z_2, \ldots, z_m$. Then*

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f, z_k) \\ &= 2\pi(\text{sum of residues of } f(z) \text{ at singularities inside of } \gamma)\end{aligned}$$

□

We have a nice formula for the residues of $f(z)$ in some special cases.

Theorem 2 (Formula for Residues at Simple Poles)**.** *Let $f(z)$ be of the form*

$$f(z) = \frac{g(z)}{h(z)}$$

where $g(z)$ and $h(z)$ are analytic in a disk about z_0 and

$$h(z_0) = 0, \qquad h'(z_0) \neq 0.$$

Then the residue of $f(z)$ at z_0 is

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

Problem 2. (10 points) Compute $\operatorname{Res}(f, z_0)$ in the following cases:

(a) $f(z) = \dfrac{e^{z^2}}{z^2+4}$ and $z_0 = 2i$.

(b) $f(z) = \tan(z)$ and $z_0 = \dfrac{\pi}{2}$

(c) $f(z) = \dfrac{g(z)}{z-a}$ where $g(z)$ is analytic and $z_0 = a$. □

Problem 3. (10 points) Compute the following

(a) $\displaystyle\int_{|z-5|=2} \frac{e^{z^2}}{z^2+4}\,dz.$

(b) $\displaystyle\int_{|z-3i|=2} \frac{e^{z^2}}{z^2+4}\,dz.$

Problem 4. (20 points) For $R > 0$ let γ_R be the upper half of the circle $|z| = R$ and σ_r the line segment on the real axis between $-R$ and R. Let C_R be union of γ_R and σ_R, which is a closed curve and which we orient so that, as usual, we move along the curve so that inside of the curve on the left in Figure 1.

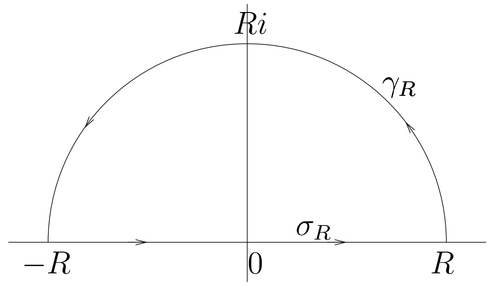

FIGURE 1. The curve γ_R is the top half of the circle $|z| = R$ and σ_R diameter of this circle between $-R$ and R. Then C_R is the union of these two curve oriented as shown.

(a) What is the length of γ_R?

Now let a and b be positive real numbers with $a \neq b$ and let

$$f(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}.$$

(b) What are the singularities of $f(z)$?
(c) Compute the resides of $f(z)$ at its singularities.
(d) Assuming that R is very large use the Residue Theorem to compute

$$\int_{C_R} f(z)\, dz.$$

□

Recall our basic estimate for the size of complex line integrals:

Proposition 3. *Let $f(z)$ be continuous on a curve γ and assume that for a positive constant M*

$$|f(z)| \leq M.$$

Then

$$\left| \int_\gamma f(z)\, dz \right| \leq M \operatorname{Length}(\gamma)$$

□

Problem 5. (20 points) Recall that the reverse triangle inequality for complex numbers is

$$|z_2 + z_1| \geq |z_2| - |z_1|.$$

(a) Use this to show that if $|z| = R$ and a and b are positive real numbers with $a, b < R$, then

$$|(z^2 + a^2)||(z^2 + b^2)| \geq (R^2 - a^2)(R^2 - b^2)$$

and therefore

$$\left| \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} \right| \leq \frac{R^2}{(R^2 - a^2)(R^2 - b^2)}.$$

(b) Let $f(z) = \dfrac{z^2}{(z^2 + a^2)(z^2 + b^2)}$ be the function of Problem 4 and also γ_R and σ_R as in Problem 4. Use part (a) of the current problem and Theorem 3 to show that when $R > a, b$ the inequality

$$\left| \int_{\gamma_R} f(z)\, dz \right| \leq \frac{\pi R^3}{(R^2 - a^2)(R^2 - b^2)}$$

holds.

(c) Show

$$\lim_{R \to \infty} \int_{\gamma_R} f(z)\, dz = 0.$$

(d) Explain why

$$\int_{\sigma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx.$$

(e) Justify the following

$$\lim_{R \to \infty} \int_{C_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}\, dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\, dx$$

(f) What is $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\, dx$?

Problem 6. (20 points) Recall that the analytic function $f(z)$ has a zero of order m at z_0 if and only if

$$f(z) = (z - z_0)^m f_1(z)$$

where $f_1(z)$ is analytic and $f_1(z_0) \neq 0$.

(a) Show

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{f_1'(z)}{f_1(z)}$$

(b) What is the residue of the function $\dfrac{f'(z)}{f(z)}$ at $z = z_0$?

(c) Let γ be a simple closed curve and assume that $f(z)$ is analytic on and inside of γ. Also assume that $f(z) \neq 0$ on γ (but $f(z)$ may have zeros inside of γ.) Use the Residue Theorem to explain why

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{number of zeros of } f(z) \text{ inside } \gamma \text{ counted with multiplicity.}$$

To be more explicit let $z_1, z_2, \ldots, z_k$ be the zeros of $f(z)$ inside of γ and let m_j be the order of z_j. Then you are to show

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = m_1 + m_2 + \cdots + m_k.$$

□

Definition 4. Let $f(z)$ have an isolated singularity at $z = z_0$ and m a positive integer. Then $z = z_0$ is a ***pole of order*** m of $f(z)$ if and only if the Laurent series of $f(z)$ is of the form

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{-m+1}} + \cdots = \sum_{n=-m}^{\infty} c_n (z-z_0)^n$$

with $c_{-m} \neq 0$. □

Thus if $f(z)$ has a pole of order 3 at z_0, then its Laurent expansion starts off as

$$f(z) = \frac{c_{-3}}{(z-z_0)^3} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

with $c_{-3} \neq 0$.

Problem 7. (10 points) Let $f(z)$ be of the form

$$f(z) = \frac{g(z)}{h(z)}$$

where $g(z)$ and $h(z)$ are analytic in disk about z_0 and so that $g(z_0) \neq 0$ and $h(z)$ has a zero of order m at z_0. Show that $f(z)$ has a pole of order m at z_0. *Hint:* As $h(z)$ has a zero of order m at z_0 you can write

$$h(z) = (z-z_0)^m h_1(z)$$

where $h_1(z)$ is analytic and $h_1(z_0) \neq 0$. Then

$$f(z) = \frac{g(z)}{(z-z_0)^m h_1(z)} = \frac{1}{(z-z_0)^m}\left(\frac{g(z)}{h_1(z)}\right) = \frac{f_1(z)}{(z-z_0)^m}$$

where

$$f_1(z) = \frac{g(z)}{h_1(z)}.$$

Explain why $f_1(z)$ is analytic in a disk about z_0 and why $f(z_0) \neq 0$. Use that $f_1(z)$ has a Taylor expansion about z_0 to finish the proof. □