Math 554 Test 1, answer key

Problem 1. Find the sum of the series $\displaystyle\sum_{k=0}^{9} \frac{3(-1)^k x^{2k}}{10^k}$. □

Solution. This is a geometric series, thus its sum is

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{\dfrac{3(-1)^0 x^0}{10^0} - \dfrac{3(-1)^{10} x^{20}}{10^{10}}}{1 - \dfrac{-x^2}{10}} \\ &= \frac{10^{10}\left(\dfrac{3(-1)^0 x^0}{10^0} - \dfrac{3(-1)^{10} x^{20}}{10^{10}}\right)}{10^{10}\left(1 - \dfrac{-x^2}{10}\right)} \\ &= \frac{3 \cdot 10^{10} - 3x^{20}}{10^{10} + 10^9 x^2}. \end{aligned}$$

□

Problem 2. Let $x_0, x_1, \dots x_{100}$ be real numbers such that

$$|x_k - x_{k-1}| < \frac{1}{2^k} \qquad \text{for} \qquad k = 1, 2, \dots, 100.$$

Show

$$|x_{100} - x_0| < 1.$$

Hint: Note that by the adding and subtracting trick and the triangle inequality we have

$$\begin{aligned} |x_0 - x_5| &= |(x_0 - x_1) + (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + (x_4 - x_5)| \\ &\le |x_0 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_4| + |x_4 - x_5| \\ &\le \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}. \end{aligned}$$

Something like this works with 5 replaced by 100. □

Solution. We use the adding and subtracting and summation notation. First note

$$\begin{aligned} x_0 - x_{100} &= (x_0 - x_1) + (x_1 - x_2) + (x_2 - x_3) + \cdots + (x_{98} - x_{99}) + (x_{99} - x_{100}) \\ &= \sum_{k=1}^{100} (x_k - x_{k-1}) \end{aligned}$$

Therefore

$$\begin{aligned}
|x_0 - x_{100}| &= \left|\sum_{k=1}^{100}(x_k - x_{k-1})\right| \\
&\le \sum_{k=1}^{100} |x_k - x_{k-1}| && \text{(triangle inequality)} \\
&\le \sum_{k=1}^{100} \frac{1}{2^k} && \text{(given)} \\
&= \frac{\frac{1}{2} - \frac{1}{2^{101}}}{1 - \frac{1}{2}} && \text{(sum of geomatic series)} \\
&< \frac{\frac{1}{2} - 0}{1 - \frac{1}{2}} \\
&= 1
\end{aligned}$$

as required. □

Problem 3. Let $b > 1$. Show that the subset $B := \{b^k : k \in \mathbb{N}\} = \{b, b^2, b^3, \ldots\}$ is unbounded in $\mathbb{R}$. *Hint:* Towards a contradiction assume that B has an upper bound. Then by the least upper bound axiom B has a least upper bound $\beta = \sup(B)$. Use this fact to derive a contradiction. □

Solution. Towards a contradiction assume that the set B is bounded. Then by the Least Upper bound axiom it will have a least upper bound, $\beta = \sup(B)$. For any natural number n the number $n + 1$ is also a natural number, thus $b^{n+1} \in B$ and as β is an upper bound for B this implies

$$b^{n+1} \le \beta$$

and dividing by b implies

$$b^n \le \frac{\beta}{b} < \beta$$

(where $\beta/b < \beta$ because $b > 1$). This implies b/β is a upper bound for B which is less than the least upper bound, a contradiction. □

Problem 4. Let A and B be subsets of $\mathbb{R}$ which are bounded above. Let

$$\alpha = \sup(A), \qquad \beta = \sup(B).$$

and let $A + B$ be the set

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove

$$\sup(A + B) = \alpha + \beta.$$

□

Solution (brute force). We first show $\alpha + \beta$ is an upper bound for $A + B$. Let $s \in A + B$, then for some $a \in A$ and $b \in B$ we have $s = a + b$. Then as α is an upper bound for A and β is an upper bound for B we have $a \leq \alpha$ and $b \leq \beta$ and thus

$$s = a + b \leq \alpha + \beta.$$

As s was any element of S this shows $\alpha + \beta$ is an upper bound for $A + B$.

To see that $\alpha + \beta$ is a least upper bound for $A + B$, let γ be any upper bound and we will show $\alpha + \beta \leq \gamma$. Let $\varepsilon > 0$. Then by the definition of least upper bound there are $a_1 \in A$ and $b_1 \in B$ such that the inequalities

$$\alpha - \varepsilon < a_1 \leq \alpha \qquad \text{and} \qquad \beta - \varepsilon < b_1 \leq \beta.$$

Then

$$\alpha + \beta - 2\varepsilon = (\alpha - \varepsilon) + (\beta - \varepsilon) < a_1 + b_2 \leq \gamma$$

where $a_1 + b_1 < \gamma$ as $a_1 + b_2 \in A + B$ and γ is an upper bound for $A + B$. Thus

$$\alpha + \beta - 2\varepsilon < \gamma$$

for all $\varepsilon > 0$, which implies $\alpha + \beta \leq \gamma$. Therefore $\alpha + \beta$ is an upper bound for $A + B$ that is $\leq \gamma$ for any other upper bound of $A + B$. Whence $\sup(A + B) = \alpha + \beta$. □

Solution (less work, but not as transparent). We start the same, let $x \in A + B$. Then $x = a + b$ for some $a \in A$ and $b \in B$. Then $a \leq \alpha$ and $b \leq \beta$ as α is an upper bound for A and β is an upper bound for B. Thus

$$x = a + b \leq \alpha + \beta$$

and so $\alpha + \beta$ is an upper bound for $A + B$ which implies

$$\sup(A + B) \leq \alpha + \beta.$$

We still have to show it is the least upper bound. Let $a \in A$ and $b \in B$. Then

$$a + b \leq \sup(A + B).$$

Rearrange this as

$$a \leq \sup(A + B) - b.$$

This shows that $\sup(A + B) - b$ is an upper bound for A for any $b \in B$ and thus

$$\alpha = \sup(A) \leq \sup(A + B) - b.$$

Rearrange this as

$$b \leq \sup(A + B) - \alpha$$

and as this works for all $b \in B$ we have that $\sup(A + B) - \alpha$ is an upper bound for B and thus

$$\beta \leq \sup(A + B) - \alpha.$$

This rearranges as

$$\alpha + \beta \leq \sup(A + B).$$

But we have already seen that $\sup(A + B) \leq \alpha + \beta$ and therefore we have the required equality $\sup(A + B) = \alpha + \beta$. □

Problem 5. Give examples of

(a) A subset A of $\mathbb{R}$ with $\sup(A) = 42$, $\inf(A) = 17$, but such that A has no maximum but it does have a minimum.
(b) A set that is bounded below, but not bounded from above.
(c) Irrational numbers α and β such that sum $\alpha + \beta$ and product $\alpha\beta$ are rational.

Solution. (a) The easiest example is the half open interval $A = [17, 42)$. Then $\inf(A) = \min(A) = 17$ and $\sup(A) = 42$, but A as no maximum (for if it did it would have to be 42 which is not in the set).
(b) A natural example is $[0, \infty)$ which is bounded below bu 0, but has no upper bound.
(c) Let $\alpha = \sqrt{3}$ and $\beta = -\sqrt{3}$. Then $\alpha + \beta = 0$ which is rational and $\alpha\beta = -3$ which is also rational.

If you want a general method to find such α and β note

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

Therefore if you choose any rational numbers r and s such that the roots of

$$x^2 - rx + s$$

are irrational, then the roots α and β have $\alpha + \beta = r$ and $\alpha\beta = s$ and so give examples.

□

Problem 6. Let $\mathbb{Q}$ be the set of rational numbers and let $S = \mathbb{Q} \cap (1, \sqrt{3})$ be the set of rational numbers between 1 and $\sqrt{3}$. Find $\alpha = \inf(S)$ and $\beta = \sup(S)$ and prove your answers are correct. You can use the fact that there is a rational number between any two real numbers. □

Solution. First $\inf(S) = 1$. For any $s \in S = \mathbb{Q} \cap (1, \sqrt{3})$ we have that s is between 1 and $\sqrt{3}$ and thus $1 < s$. Therefore 1 is a lower bound for S. Let a be any lower bound for S. Towards a contradiction assume that $a > 1$. Then there is a rational number, r, between 1 and a. But then $r \in S$ as it is a rational number between 1 and $\sqrt{3}$. But r between 1 and a implies $r < a$ contradicting that a is a lower bound for S. Therefore 1 is the greatest lower bound for S.

The proof that $\sup(S) = \sqrt{3}$ is almost identical. First $\sqrt{3}$ is a upper bound as any element, s, of S has $s \le \sqrt{3}$. Towards a contradiction assume that $\sqrt{3}$ is not the least upper bound. Then there is an upper bound b of S that is less than $\sqrt{3}$. Then there is a rational number, r, between b and $\sqrt{3}$. This rational number is in S as it is rational $1 < r < \sqrt{3}$. But $r > b$ contradicting that b is an upper bound. So $\sup(S) = \sqrt{3}$. □

We have defined a function $f\colon [a, b] \to \mathbb{R}$ to be ***Lipschitz*** if and only if there is a number $M \ge 0$ such that

$$|f(x_2) - f(x_1)| \le M|x_2 - x_1|$$

for all $x_1, x_2 \in [a, b]$.

Problem 7. Let $f(x) = \sqrt{x}$ on $[0, \infty)$.

(a) Show $f(x) = \sqrt{x}$ is Lipschitz on the interval $[1, 100]$. *Hint:* One way to start this is to use just the opposite of rationalizing the denominator, which is to rationalize the numerator. A example calculation looks like

$$\begin{aligned}\sqrt{65} - \sqrt{64} &= \frac{(\sqrt{65} - \sqrt{64})(\sqrt{65} + \sqrt{64})}{(\sqrt{65} + \sqrt{64})} \\ &= \frac{(\sqrt{65})^2 - (\sqrt{64})^2}{\sqrt{65} + \sqrt{64}} \\ &= \frac{65 - 64}{\sqrt{65} + \sqrt{64}} \\ &= \frac{1}{\sqrt{65} + \sqrt{64}} \\ &< \frac{1}{\sqrt{64} + \sqrt{64}} && (\text{as } \sqrt{65} > \sqrt{64}) \\ &= \frac{1}{16}.\end{aligned}$$

(b) Show that $f(x)$ is not Lipschitz on the interval $[0, 1]$. *Hint:* Assume that $f(x)$ is Lipschitz on $[0, 1]$. The there is a constant M such that

$$|\sqrt{x_2} - \sqrt{x_1}| \le M|x_2 - x_1|$$

for all $x_1, x_2 \in [0, 1]$. Letting $x_1 = 0$ gives $\sqrt{x_2} \le M x_2$ for all $x_2 \in [0, 1]$. Show this leads to a contradiction. $\square$

Solution. (a) Let $x_1, x_2 \in [1, 100]$. Then $\sqrt{x_1}, \sqrt{x_2} \ge 1$.

$$\begin{aligned}|f(x_2) - f(x_1)| &= |\sqrt{x_2} - \sqrt{x_1}| \\ &= \left| \frac{(\sqrt{x_2} - \sqrt{x_1})(\sqrt{x_2} + \sqrt{x_1})}{\sqrt{x_2} + \sqrt{x_1}} \right| \\ &= \left| \frac{(\sqrt{x_2})^2 - (\sqrt{x_1})^2}{\sqrt{x_2} + \sqrt{x_1}} \right| \\ &= \left| \frac{x_2 - x_1}{\sqrt{x_2} + \sqrt{x_1}} \right| \\ &= \frac{|x_2 - x_1|}{\sqrt{x_2} + \sqrt{x_1}} \\ &\le \frac{|x_2 - x_1|}{1 + 1} && (\text{as } \sqrt{x_1}, \sqrt{x_2} \ge 1) \\ &= \frac{1}{2}|x_2 - x_1|.\end{aligned}$$

This show $f(x)$ is Lipschitz on $[0, 100]$.

(b) As per the hint, assume that there is a constant M such that $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$ and let $x_1 = 0$ and $x_2 = x$. Then the Lipschitz inequality implies for any x with $0 < x \leq 1$ that

$$\sqrt{x} = |\sqrt{x} - \sqrt{0}| \leq M|x - 0| = Mx.$$

Dividing by $\sqrt{x}$ gives

$$1 \leq M\sqrt{x}.$$

Letting $x = 1/(2M)^2$ in this gives

$$1 \leq M\sqrt{x} = M\sqrt{\frac{1}{(2M)^2}} = \frac{M}{2M} = \frac{1}{2}$$

which is the contradiction that completes the proof.

□

Problem 8. Let f be Lipschitz on an interval $[a, b]$ and assume that for some positive number c that $f(x) \geq c$ on $[a, b]$. Prove that $g(x)$ defined by

$$g(x) = \frac{1}{f(x)}$$

is Lipschitz on $[a, b]$. □

Solution. Since $f(x)$ is Lipschitz on $[a, b]$ there is a constant M such that

$$|f(x_2) - f(x_1)| \leq M.$$

Therefore

$$\begin{aligned}
|g(x_2) - g(x_1)| &= \left|\frac{1}{f(x_1)} - \frac{1}{f(x_2)}\right| \\
&= \left|\frac{f(x_1) - f(x_2)}{f(x_1)f(x_2)}\right| \\
&= \frac{|f(x_1) - f(x_2)|}{f(x_1)f(x_2)} \\
&\leq \frac{M|x_1 - x_2|}{f(x_1)f(x_2)} && (\text{as } |f(x_2) - f(x_1)| \leq M) \\
&\leq \frac{M|x_1 - x_2|}{c \cdot c} && (\text{as } f(x_1), f(x_2) \geq c\ ) \\
&= \frac{M}{c^2}|x_1 - x_2| \\
&= M'|x_1 - x_2|
\end{aligned}$$

where $M' = \frac{M}{c^2}$. This shows that g is Lipschitz. □

Problem 9. Let n be an odd positive integer and let $h(x)$ be a function on all of $\mathbb{R}$ that satisfies the two conditions

$$|h(x_2) - h(x_1)| \leq A|x_2 - x_1|$$
$$|h(x)| \leq B + C|x|^{n-1}$$

for some constants A, B, C and all $x_1, x_2, x \in \mathbb{R}$. Let $f(x)$ be defined by

$$f(x) = x^n + h(x).$$

(a) Explain why for any $b > 0$ that $f(x)$ is Lipschitz on $[-b, b]$. *Hint:* You can use the facts that polynomials are Lipschitz on any finite interval and that the sum of two Lipschitz functions is Lipschitz.

(b) Show that there is a $b > 0$ such that

$$f(-b) < 0, \qquad \text{and} \qquad f(b) > 0.$$

(c) Give the statement of the Lipschitz Intermediate Value Theorem and say how it implies that $f(x) = 0$ has a solution for some x with $-b < x < b$.

Solution. (a) The function x^n is a polynomial thus is Lipschitz on the interval $[-b, b]$. The function $h(x)$ is Lipschitz (as $|h(x_2)-h(x_1)| \leq A|x_2-x_1|$) and therefore $f(x) = x^n + h(x)$ is the sum of Lipschitz functions and therefore $f(x)$ is Lipschitz.

(b) Assume that $x \geq 1$. Then x is positive so $|x| = x$ and $|x|^{n-1} \geq 1$. Then

$$\begin{aligned}
f(x) &= x^n + h(x) \\
&\geq x^n - |h(x)| \\
&\geq x^n - (B + C|x|^{n-1}) && (\text{as } |h(x)| \leq B + C|x|^{n-1}) \\
&\geq x^n - (B|x|^{n-1} + C|x|^{n-1}) && (\text{as } |x|^{n-1} \geq 1) \\
&= x^n - (Bx^{n-1} + Cx^{n-1}) && (\text{as } x = |x| \text{ for } x \geq 1) \\
&= x^{n-1}\,(x - (B + C))
\end{aligned}$$

Therefore if $x \geq$ and $x > (B + C)$, we have

$$f(x) \geq x^{n-1}\,(x - (B + C)) = (\text{positive}) \times (\text{positive})$$

and thus $f(x)$ is positive. In particular $f(x) > 0$ when $x \geq 1 + B + C$.

Now assume that $x \leq -1$. Then x is negative so $|x| = -x$. As n is odd $|x|^n = (-x)^n = (-1)^n x^n = -x^n$ as $(-1)^n = -1$ for odd numbers. Likewise $n-1$ is even so a similar calculation shows $|x|^{n-1} = (-x)^{n-1} = x^{n-1}$. So for $x \leq -1$

$$\begin{aligned}
f(x) &= x^n + h(x) \\
&\leq x^n + |h(x)| \\
&\leq x^n + B + C|x|^{n-1} && (\text{as } |h(x)| \leq B + C|x|^{n-1}) \\
&\leq x^n + B|x|^{n-1} + C|x|^{n-1} && (\text{as } |x| \geq 1) \\
&= x^n + x^{n-1}(B + C) && (\text{as } x^{n-1} = |x|^{n-1}) \\
&= x^{n-1}(x + (B + C)).
\end{aligned}$$

Now choose our $x \leq -1$ so that also $x < -(B+C)$, then $x+(B+C) < 0$ and so

$$f(x) \leq x^{n-1}(x + (B + C)) = (\text{positive}) \times (\text{negative})$$

and therefore $f(x) < 0$. So if $x \leq -(1 + B + C)$, then $f(x) < 0$.

Therefore if we let $b = 1 + B + C$ we have $f(-b) < 0$ and $f(b) > 0$.

(c) The statement is

Lipschitz Intermediate Value Theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a Lipschitz function such that $f(a) < 0$ and $f(b) > 0$. Then there is a point $\xi \in (a,b)$ with $f(\xi) = 0$. That is $f(x) = 0$ has a solution with for some x between a and b.*

In our case we have the Lipschitz function $f\colon [-b,b] \to \mathbb{R}$ with $f(-b) < 0$ and $f(b) > 0$ so there is a $x = \xi$ with $-b < \xi < b$ $f(x) = 0$. □

Problem 10. Let

$$h(x) = ax^3 + bx^2 + cx + d$$

be a cubic polynomial. Show there are constants B and C such that

$$|h(x)| \leq B + C|x|^3$$

for all $x \in \mathbb{R}$. □

Solution. First assume $|x| \leq 1$. Then

$$\begin{aligned} |h(x)| &= |ax^3 + bx^2 + cx + d| \\ &\leq |a||x|^3 + |b||x|^2 + |c||x| + |d| && \text{(triangle inequality)} \\ &\leq |a| + |b| + |c| + |d| && \text{(as } |x| \leq 1) \end{aligned}$$

For $|x| \geq 1$ we have $|x|^3 \geq |x|^2$, $|x|^3 \geq |x|$ and $|x|^3 \geq 1$ and thus

$$\begin{aligned} |h(x)| &= |ax^3 + bx^2 + cx + d| \\ &\leq |a||x|^3 + |b||x|^2 + |c||x| + |d| && \text{(triangle inequality)} \\ &\leq |a||x|^3 + |b||x|^3 + |c||x|^3 + |d||x|^3 && \text{(as } |x| \geq 1) \\ &= (|a| + |b| + |c| + |d|)|x|^3 \end{aligned}$$

Therefore the inequality

$$|h(x)| \leq (|a| + |b| + |c| + |d|)|x|^3 + (|a| + |b| + |c| + |d|)$$

holds for all x. Thus if we set $B + C = |a| + |b| + |c| + |d|$ we have

$$|h(x)| \leq B|x|^3 + C$$

as required. □