Math 554 Test 2, Answer Key.

Problem 1. Let E be a metric space and let $a \in E$. Define a function $f_a \colon E \to \mathbb{R}$ by
$$f(p) = d(p, a).$$
Prove that f satisfies
$$|f(p) - f(q)| \le d(p, q).$$
□

Solution. This is not much more than a restatement of the reverse triangle inequality:
$$|f(p) - f(q)| = |d(a, p) - d(a, q)| \le d(p, q).$$
This was enough to do the problem, but in case you also want to see the proof of the reverse triangle inequality again, here it is. First two application of the triangle inequality
$$d(a, p) \le d(a, q) + d(q, p)$$
$$d(a, q) \le d(a, p) + d(p, q).$$
These can be rearranged into
$$-d(p, q) \le d(a, p) - d(a, q) \le d(p, q)$$
which implies
$$|d(a, p) - d(a, q)| \le d(p, q),$$
as, in general, $-A \le x \le A$ implies $|x| \le A$. □

Recall that a function $f \colon E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant M such that
$$|f(p) - f(q)| \le M d(p, q)$$
for all $p, q \in E$. Thus Problem 1 tells us that the function $f(p) = d(a, p)$ is Lipschitz with $M = 1$.

We have also proven

Proposition 1. *If $f \colon E \to \mathbb{R}$ is Lipschitz and $\langle p_n \rangle_{n=1}^{\infty}$ is a convergent sequence, say*
$$\lim_{n\to\infty} p_n = p$$
then
$$\lim_{n\to\infty} f(p_n) = f(p).$$

Problem 2. As a bit of review give the proof of this. That is show for all $\varepsilon > 0$ there is a N such that

$$n \geq N \qquad \text{implies} \qquad |f(p) - f(p_n)| < \varepsilon.$$

□

Solution. As f is Lipschitz there is a $M > 0$ such that

$$|f(p) - f(q)| \leq M d(p, q).$$

Let $\varepsilon > 0$ By the definition of $\lim_{n\to\infty} p_n = p$, there is a N such that

$$n \geq N \quad \implies \quad d(p_n, p) < \frac{\varepsilon}{M}.$$

Then $n \geq N$ implies

$$\begin{aligned} |f(p_n) - f(p)| &\leq M d(p, p_n) \\ &< M \frac{\varepsilon}{M} \\ &= \varepsilon \end{aligned}$$

which verifies the definition of $\lim_{n\to\infty} f(p_n) = f(p)$. □

Definition 2. Let E be a metric space. Then a subset S of E is ***sequentially compact*** if and only if for every sequence of points $\langle p_n \rangle_{n=1}^{\infty} \subseteq S$ from S there is a subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ such that for some point $p \in S$

$$p = \lim_{k\to\infty} p_{n_k}.$$

□

Problem 3. Let E be a metric space and $S \subseteq E$ a sequentially compact subset of E. Let $a \in E$. Show there is a point $p \in S$ such that

$$d(a, p) \leq d(a, q) \qquad \text{for all} \qquad q \in S.$$

(That is there is a point of S that is closest to a.) *Hint:* One way to do this is by using the following steps:

(a) Let D be the set of distances

$$D = \{d(a, q) : q \in S\}.$$

Show that this is a subset of $\mathbb{R}$ that is bounded below and thus D has an infimum (greatest lower bound). Let

$$\beta = \inf(D).$$

Also give a sentence saying why

$$\beta \leq d(a, q) \qquad \text{for all} \qquad q \in S.$$

Solution. If $y \in D$, then $y = d(a, q) \geq 0$ for some $q \in S$ and therefore D is bounded below by 0. Whence

$$\beta = \inf(D)$$

exists by the greatest lower bound property of the real numbers. Since all elements of D are of the form $d(a, q)$ for some $q \in S$ and β is a lower bound for S the inequality

$$\beta \leq d(a, q)$$

holds for all $q \in S$. □

(b) Explain why for each positive integer n there is a point $p_n \in S$ with

$$\beta \leq d(a, p_n) < \beta + \frac{1}{n}.$$

Solution. By the definition of $\inf(D)$ as the greatest lower bound of D we have that for any $x > \beta$ there is an element $y \in D$ with

$$\beta \leq y < x.$$

In particular letting $x = \beta + 1/n > \beta$ implies there is an element $y_n \in D$ with $\beta \leq y_n < \beta + 1/n$. By the definition of D we see that $y_n = d(a, p_n)$ for some $p_n \in S$. Therefore

$$\beta \leq d(a, p_n) < \beta + \frac{1}{n}$$

□

(c) Explain why $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence that converges to a point $p \in S$, that is

$$\lim_{k \to \infty} p_{n_k} = p \in S.$$

Solution. Not much to do here. By the definition of sequentially compact such a subsequence exists. □

(d) Show that

$$\lim_{k \to \infty} d(a, p_{n_k}) = d(a, p).$$

(*Subhint:* Use Problem 1 and Proposition 1.)

Solution. As per the subhint, let f be defined on E by

$$f(q) = d(a, q).$$

By Problem 1 this is Lipschitz (with $M = 1$) and therefore by Problem 2,

$$\lim_{k \to \infty} f(p_{n_k}) = f(p).$$

Rewriting in terms of the definition of f gives

$$\lim_{k\to\infty} d(a, p_{n_k}) = d(a, p)$$

as required. □

(e) Use Part (b) of the problem to also show that

$$\lim_{k\to\infty} d(a, p_{n_k}) = \beta.$$

Solution. Let $\varepsilon > 0$ and let N be a positive integer with

$$\frac{1}{N} < \varepsilon.$$

As for any subsequence $n_k \geq k$ and part (b) we have for $k \geq N$

$$\beta \leq d(a, p_{n_k}) < \beta + \frac{1}{n_k} \leq \beta + \frac{1}{N} < \beta + \varepsilon.$$

This implies that for $k \geq N$

$$|\beta - d(p, p_{n_k})| < \varepsilon$$

which is the definition of $\lim_{k\to\infty} d(a, p_{n_k}) = \beta$. □

(f) Now finish the proof by showing $d(a, p) = \beta$ and explaining why this shows $d(a, p) \leq d(a, q)$ for all $q \in S$.

Solution. Combining parts (d) and (e) we have

$$d(a, p) = \lim_{k\to\infty} d(a, p_{n_k}) = \beta.$$

Then

$$d(a, p) = \beta \leq d(a, q)$$

for all $q \in S$ follows from part (a). □

Problem 4. Let E be a metric space and let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in E, say

$$\lim_{n\to\infty} p_n = p.$$

Let

$$S = \{p\} \cup \{p_n : n = 1, 2, 3, \ldots\}.$$

Show S is closed and bounded. □

Solution. We first show that S is closed. It is enough to show the compliment, $\mathcal{C}S$, is open. Let $q \in \mathcal{S}$. Then we wish to show there is a $r > 0$, so that $B(q, r) \subseteq \mathcal{S}$ where $B(q, r)$ is the ball of radius r about q. Restated we wish to show that there is $r > 0$ so that $B(q, r) \cap S = \varnothing$. Towards a contradiction assume this is not the case. Note if we choose $r < d(p, q)$ we have that $q \notin B(q, r)$ and therefore we can, by making r smaller, assume that each $B(q, r)$ contains a point of S other than p, that is a point of the form p_n for some n. Then for any positive integer k and $r = 1/k$ we have a point $p_{n_k} \in B(q, 1/k)$. That is

$$d(q, p_{n_k}) < \frac{1}{k}.$$

This implies

$$\lim_{k\to\infty} p_{n_k} = q.$$

But $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence of the original sequence and so we have

$$\lim_{k\to\infty} p_{n_k} = \lim_{n\to\infty} p_n = p$$

which implies $p = q$, contradicting that $q \notin S$.

We have proven in class that Cauchy sequences are bounded and that convergent sequences are Cauchy, so we could just quote those facts to conclude that S is bounded. Here is direct proof. Using $\varepsilon = 1$ in the definition of $\lim_{n\to\infty} p_n = p$ we find a positive integer N such that

$$n \geq N \qquad \Longrightarrow \qquad d(p, p_n) < 1.$$

Let

$$r = \max\{1, d(p, p_1), d(p, p_2), \ldots, d(p, p_N)\}$$

The every point of S is in the closed ball $\overline{B}(p, r)$ which implies S is bounded. □

Here is anther of our recent results:

Theorem 3. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in a metric space such that it has a convergent subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.* □

Problem 5. Let E be a metric space were every closed bounded set is sequentially compact. Show E is complete. *Hint:* To show E is complete we need to show that every Cauchy sequence in E converges. So let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E and let S be defined as in Problem 4. By that problem S is closed and bounded. Now explain why the assumption on sequential compactness and Theorem 4 implies the sequence converges. □

Solution. Towards a contradiction assume that there is a Cauchy sequence $\langle p_n \rangle_{n=1}^{\infty}$ that does not converge. Let

$$S = \{p_n, n = 1, 2, 3, \ldots\}$$

Then S is bounded be a result we have proven in class (Proposition 3.49, Page 63 of *Notes on Analysis*). The same argument we gave for Problem 4 implies that S is closed. Therefore S is closed and bounded and thus by assumption sequentially compact. This implies there is a subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$ of S that converges to a point of S. But by Theorem 3 above this implies the original sequence $\langle p_n\rangle_{n=1}^{\infty}$ converges, a contradiction. □

The following may be useful in the next problem.

Theorem 4. *A bounded monotone sequence in $\mathbb{R}$ is convergent.* □

Problem 6. Define a sequence in $\mathbb{R}$ by

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \frac{3}{4}x_1 + 12 \\ x_3 &= \frac{3}{4}x_2 + 12 \\ x_4 &= \frac{3}{4}x_3 + 12 \end{aligned}$$

and in general

$$x_n = \frac{3}{4}x_{n-1} + 12.$$

(a) Compute x_2, x_3, and x_4.

Solution.

$$\begin{aligned} x_2 &= \frac{51}{4} = 12.75 \\ x_3 &= \frac{345}{16} = 21.5625 \\ x_4 &= \frac{1083}{64} = 28.171875 \end{aligned}$$

□

(b) Use induction to show the sequence $\langle x_n\rangle_{n=1}^{\infty}$ is increasing. That is show $x_n > x_{n-1}$ for all $n \geq 2$.

Solution. The base of the induction is

$$x_1 = 1 < 12.75 = x_2.$$

Assume that $x_{n-1} < x_n$ (the induction hypothesis). We wish to use this to prove $x_n < x_{n+1}$. The calculation is

$$x_{n+1} = \frac{3}{4}x_n + 12 > \frac{3}{4}x_{n-1} + 12 = x_n.$$

This completes the induction. □

(c) Use induction to show $x_n \leq 100$ for all n.

Solution. The base base is $x_1 = 1 < 100$. Assume $x_n < 100$ (the induction hypothesis) and we wish to use this to prove $x_{n+1} < 100$ The calculation is

$$x_{n+1} = \frac{3}{4}x_n + 12 < \frac{3}{4}x_n + 12 < \frac{3}{4}(100) + 12 = 87 < 100.$$

□

(d) Show $\langle x_n \rangle_{n=1}^{\infty}$ converges and find its limit. □

Solution. The sequence is bounded and increasing and therefore convergent by Theorem 4. Let

$$x = \lim_{n\to\infty} x_n$$

be its limit. Then we also have

$$\lim_{n\to\infty} x_{n-1} = x$$

Whence taking the limit as $n \to \infty$ in

$$x_n = \frac{3}{4}x_{n-1} + 12$$

gives

$$x = \frac{3}{4}x + 12$$

Solving this for x gives

$$\lim_{n\to\infty} x_n = x = 48.$$

□

Problem 7. Show that the subset S be the subset of $\mathbb{R}^2$ defined by

$$S = \{(x, y) : 0 < x^2 + y^2 \leq 1\}.$$

(a) Draw a picture of S.

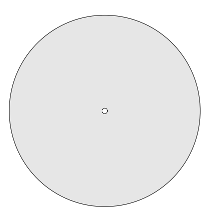

FIGURE 1. This is the closed unit disk centered at the origin with $(0, 0)$ omitted.

Solution. □

(b) Show that S is not sequentially compact.

Solution. We need to find a sequence from S that does not have any subsequence that converges to a point of S. Let

$$p_n = \left(\frac{1}{n}, 0\right)$$

For $n = 1, 2, 3, \ldots$ each of these points is in S and

$$\lim_{n\to\infty} p_n = (0,0) \notin S.$$

Any subsequence of $\langle p_n \rangle_{n=1}^{\infty}$ will also converge to $(0,0)$ and thus does not converge to a point of S. Therefore S is not sequentially compact. $\square$