Math 554, Final.

- *This is due on Thursday, April 29 by midnight. It should be submitted via Blackboard as a pdf document and should have your name on the first page.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *You will be graded in part on writing proofs up correctly. In particular you can lose points with answers that are all formulas and equations without any English.*

We started the class out with some review of some algebra that is useful. One example of this was summing a finite geometric series

$$\sum_{k=0}^{n} ar^k = a + ar + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r} = \frac{\textsf{first} - \textsf{next}}{1 - \textsf{ratio}}.$$

which holds for $r \neq 0$ and the related formulas for factoring the difference of powers

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = (x-y)\sum_{i+j=n-1} x^i y^j$$

Problem 1. (15 points) Assume that on January first you put one cent (i.e. 1¢) in a jar, on the second of January you put 2¢ in the jar, on January 3 you put 4¢ in the jar, on January 4 you put 8¢ in the jar and continue doubling the number of cents you put in the jar each day up to and including January 31.

(a) How many pennies do you put into the jar on the n-th day? That is how many do you put into the jar on January n?

(b) What is the total value (in dollars) of the pennies in the jar at the end of the month? □

Problem 2. (10 points) Let

$$f(x) = 3x^3 + x$$

and assume $|x| < 4$ and $|y| < 5$. Show

$$|f(x) - f(y)| \leq 184|x - y|.$$

The other important formula we reviewed was the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Problem 3. (10 points) Find a simple formula for the sum

$$\sum_{k=0}^{n} \binom{n}{k} 3^k 2^{n-k}$$

□

We then defined the real numbers $\mathbb{R}$ as the unique field (a set with operations of addition and multiplication that satisfy the usual rules of algebra and where subtraction and division are defined) with an ordering $<$ and such that bounded sets have least upper bounds. Explicitly the property of the real numbers that makes most of our analysis theorems hold is

Least Upper Upper Bound Property. *Let S be nonempty subset of $\mathbb{R}$ that is bounded above. Then S has* ***least upper bound****, denoted* $\sup(S)$*. This is the unique real number such that* $\sup(S)$ *is an upper bound for S and* $\sup(S) \leq b$ *for any other upper bound of S.* □

Problem 4. (20 points) Give examples of

(a) A subset $S \subseteq \mathbb{R}$ such $\sup(S) = 42$, but $42 \notin S$.
(b) A subset $S \subseteq \mathbb{R}$ such that $\sup(S) = 17$ and $17 \in S$.
(c) A set $S \subseteq \mathbb{R}$ such that every element of S is a rational number, but $\sup(S)$ is irrational. □

Problem 5. (15 points) Use the triangle inequality

$$|a+b| \leq |a| + |b|$$

and the reverse triangle inequality

$$|a+b| \geq |a| - |b|$$

to show that if $|x| \geq 2$ then

$$x^2 - 1 \geq 3$$

and

$$\frac{x^2+1}{x^2-1} \leq \frac{5}{3}.$$

Hint: It might be useful to first show

$$\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1}$$

holds. □

Our next major topic was the definition and basic properties of metric spaces. Definitions you should definitely know are that of open and closed set, adherent points and the definition of the limit of a sequence in a metric space.

Problem 6. (10 points) Let E be a metric space and S and F subsets of E with

$$S \subseteq F.$$

Show that if p is an adherent point of S, then p is also an adherent of F. $\square$

One of our basic results is that the intersection of any collection of closed subsets is closed:

Proposition 1. *Let E be a metric space with distance function $d(x, y)$. Let F_α be closed subsets of E for all $\alpha \in I$. Then then the intersection*

$$F = \bigcap_{\alpha \in I} F_\alpha$$

is a closed set. $\square$

Definition 2. Let E be a metric space and $S \subseteq E$. Let

$$\mathcal{F} = \{F : F \text{ is closed and } S \subseteq F\}$$

(that is $\mathcal{F}$ is the collection of all closed subsets that contain S). Then the ***closure*** of S is

$$\overline{S} = \bigcap_{F \in \mathcal{F}} F.$$

(By Proposition 1 this is a closed set as it is the intersection of a collection of closed sets). $\square$

Problem 7. (20 points) Let E be a metric space and $S \subseteq E$. Show that $p \in \overline{S}$ if and only if p is an adherent point of S by showing the following

(a) If p is an adherent point of S, then $p \in \overline{S}$. *Hint:* By Problem 6 if F is a closed set with with $S \subseteq F$, then p is an adherent point F. Use some results we have covered during the term to explain why $p \in F$. Now use this to show $p \in \overline{S}$.

(b) If p is not an adherent point of S, then $p \notin \overline{S}$. *Hint:* As p is not an adherent point of S, then there is $r > 0$ so that $B(p, r) \cap S = \varnothing$. Then the compliment $\mathcal{C}(B(p, r))$ is a closed set with $S \subseteq \mathcal{C}(B(p, r))$ and $p \notin \mathcal{C}(B(p, r))$. Explain why this shows $p \notin \overline{S}$. $\square$

Problem 8. (10 points) Let E be a metric space and $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Let $\langle q_n \rangle_{n=1}^{\infty}$ be anther sequence in E with

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Show that $\langle q_n \rangle_{n=1}^{\infty}$ is also a Cauchy sequence.

Anther important property of metric spaces is completeness.

Definition 3. A metric space E is ***complete*** if and only if every Cauchy sequence $\langle p_n \rangle_{n=1}^{\infty}$ in E converges. □

We then used the Least Upper Bound property so show the real number $\mathbb{R}$ and the spaces $\mathbb{R}^n$ are complete.

Anther basic property of subsets of a metric space is compactness. Our basic general result is

Theorem 4. *A subset K is a metric space E is compact (open cover definition) if and only if it is sequentially compact (sequences have subsequences that converge to a point of the set).* □

Problem 9. (10 points) Let E be a compact metric space. Show E is complete. □

Problem 10. (15 points) Let S be the subset of $\mathbb{R}^2$ defined by

$$S = \{(x, y) : x^2 + y^2 = 1\}$$

(a) Show that S is a closed subset of $\mathbb{R}^2$. *Hint:* The function $f(x, y) = x^2 + y^2$ is continuous as it is a polynomial. Also

$$S = \{(x, y) : f(x, y) = 1\} = f^{-1}[\{1\}].$$

You should now be able to quote a result about continuous functions to say why S is closed.

(b) Show that S is bounded. Therefore S is a closed bounded subset of $\mathbb{R}^2$ and whence compact. □

You should also look over the definition of a subset of a metric space being connected. Our basic examples of connected sets are given in

Theorem 5. *(a) The connected subsets of the real numbers $\mathbb{R}$ are the intervals.*

(b) Any starlike subset of $\mathbb{R}^n$ is connected. (See page 87 of Notes on Analysis *for the definition of starlike.)* □

The last major topic we covered was continuous functions between metric spaces.

We proved the general form of the Intermediate Value Theorem which is that the continuous image of a connected set is connected. Combined with the fact that intervals in $\mathbb{R}$ are connected this gives

Theorem 6. *If $f\colon E \to \mathbb{R}$ is continuous and E is connected and f changes sign on E (that is takes on both positive and negative values) then there is a point $p \in E$ with $f(p) = 0$.* □

In what follows you can assume the following:

Proposition 7. *The circle*

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is connected. □

Proposition 8. *Let S be the circle just defined and let $f\colon S \to \mathbb{R}$ be continuous. Show there is a point $(x_0, y_0) \in S$ such that*

$$f(x_0, y_0) = f(-x_0, -y_0).$$

Problem 11. (15 points) Prove this along the following lines:

(a) Define a new function $g\colon S \to \mathbb{R}$ by

$$g(x, y) = f(x, y) - f(-x, -y)$$

and prove g is continuous.

(b) Show that

$$g(-x, -y) = -g(x, y)$$

and use this to explain why g changes sign.

(c) Now use the Intermediate Value Theorem to show that there is a point $(x_0, y_0) \in S$ with $g(x_0, y_0) = 0$ and explain why this finishes the proof. □

Have a good summer.