Math 554 Test 1.

- *This is due on Tuesday, February 16 by midnight. It should be submitted via Blackboard as a pdf document.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *Please print your name on the first page of the test.*
- *Since you have plenty of time on this test you should submit neat papers. By this I do not mean handwriting, but more not having crossed out work on your exam and also taking the time to write sentences explaining what you are doing. If you are writing the paper by hand, it is good idea to make a rough draft to get the details correct before making the final copy.*
- *Related to the last point, use common sense about simplification. Leaving a fraction as* $\frac{15}{20}$ *(rather than* $\frac{3}{4}$*), or worse yet* $\frac{6}{2}$ *(rather than* 3*) does not make a good impression. Likewise* $\cos(\pi/4)$ *is* $\sqrt{2}/2$ *etc.*

Problem 1. (5 points) Find the sum of the series $\displaystyle\sum_{k=0}^{9} \frac{3(-1)^k x^{2k}}{10^k}$. $\square$

Problem 2. (5 points) Let $x_0, x_1, \ldots x_{100}$ be real numbers such that

$$|x_k - x_{k-1}| < \frac{1}{2^k} \qquad \text{for} \qquad k = 1, 2, \ldots, 100.$$

Show

$$|x_{100} - x_0| < 1.$$

Hint: Note that by the adding and subtracting trick and the triangle inequality we have

$$\begin{aligned} |x_0 - x_5| &= |(x_0 - x_1) + (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + (x_4 - x_5)| \\ &\le |x_0 - x_1| + |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_4| + |x_4 - x_5| \\ &\le \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}. \end{aligned}$$

Something like this works with 5 replaced by 100. $\square$

Problem 3. (5 points) Let $b > 1$. Show that the subset $B := \{b^k : k \in \mathbb{N}\} = \{b, b^2, b^3, \ldots\}$ is unbounded in $\mathbb{R}$. *Hint:* Towards a contradiction assume that B has an upper bound. Then by the least upper bound axiom B has a least upper bound $\beta = \sup(B)$. Use this fact to derive a contradiction. $\square$

Problem 4. (10 points) Let A and B be subsets of $\mathbb{R}$ which are bounded above. Let

$$\alpha = \sup(A), \qquad \beta = \sup(B).$$

and let $A + B$ be the set

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Prove

$$\sup(A + B) = \alpha + \beta. \qquad \square$$

Problem 5. (15 points) Give examples of

(a) A subset A of $\mathbb{R}$ with $\sup(A) = 42$, $\inf(A) = 17$, but such that A has no maximum but it does have a minimum.
(b) A set that is bounded below, but not bounded from above.
(c) Irrational numbers α and β such that sum $\alpha + \beta$ and product $\alpha\beta$ are ration.

Problem 6. (10 points) Let $\mathbb{Q}$ be the set of rational numbers and let $S = \mathbb{Q} \cap (1, \sqrt{3})$ be the set of rational numbers between 1 and $\sqrt{3}$. Find $\alpha = \inf(S)$ and $\beta = \sup(S)$ and prove your answers are correct. You can use the fact that there is a rational number between any two real numbers. $\square$

We have defined a function $f\colon [a, b] \to \mathbb{R}$ to be ***Lipschitz*** if and only if there is a number $M \geq 0$ such that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

for all $x_1, x_2 \in [a, b]$.

Problem 7. (10 points) Let $f(x) = \sqrt{x}$ on $[0, \infty)$.

(a) Show $f(x) = \sqrt{x}$ is Lipschitz on the interval $[1, 100]$. *Hint:* One way to start this is to use just the opposite of rationalizing the denominator, which is to rationalize the numerator. A example calculation looks like

$$\begin{aligned}
\sqrt{65} - \sqrt{64} &= \frac{(\sqrt{65} - \sqrt{64})(\sqrt{65} + \sqrt{64})}{(\sqrt{65} + \sqrt{64})} \\
&= \frac{(\sqrt{65})^2 - (\sqrt{64})^2}{\sqrt{65} + \sqrt{64}} \\
&= \frac{65 - 64}{\sqrt{65} + \sqrt{64}} \\
&= \frac{1}{\sqrt{65} + \sqrt{64}} \\
&< \frac{1}{\sqrt{64} + \sqrt{64}} && (\text{as } \sqrt{65} > \sqrt{64}) \\
&= \frac{1}{16}.
\end{aligned}$$

(b) Show that $f(x)$ is not Lipschitz on the interval $[0, 1]$. *Hint:* Assume that $f(x)$ is Lipschitz on $[0, 1]$. The there is a constant M such that

$$|\sqrt{x_2} - \sqrt{x_1}| \leq M|x_2 - x_1|$$

for all $x_1, x_2 \in [0, 1]$. Letting $x_1 = 0$ gives $\sqrt{x_2} \leq Mx_2$ for all $x_2 \in [0, 1]$. Show this leads to a contradiction. $\square$

Problem 8. (10 points) Let f be Lipschitz on an interval $[a, b]$ and assume that for some positive number c that $f(x) \geq c$ on $[a, b]$. Prove that $g(x)$ defined by

$$g(x) = \frac{1}{f(x)}$$

is Lipschitz on $[a, b]$. $\square$

Problem 9. (20 points) Let n be an odd positive integer and let $h(x)$ be a function on all of $\mathbb{R}$ that satisfies the two conditions

$$|h(x_2) - h(x_1)| \leq A|x_2 - x_1|$$
$$|h(x)| \leq B + C|x|^{n-1}$$

for some constants A, B, C and all $x_1, x_2, x \in \mathbb{R}$. Let $f(x)$ be defined by

$$f(x) = x^n + h(x).$$

(a) Explain why for any $b > 0$ that $f(x)$ is Lipschitz on $[-b, b]$. *Hint:* You can use the facts that polynomials are Lipschitz on any finite interval and that the sum of two Lipschitz functions is Lipschitz.
(b) Show that there is a $b > 0$ such that

$$f(-b) < 0, \qquad \text{and} \qquad f(b) > 0.$$

(c) Give the statement of the Lipschitz Intermediate Value Theorem and say how it implies that $f(x) = 0$ has a solution for some x with $-b < x < b$.

Problem 10. (5 points) Let

$$h(x) = ax^3 + bx^2 + cx^+d$$

be a cubic polynomial. Show there are constants B and C such that

$$|h(x)| \leq B + C|x|^3$$

for all $x \in \mathbb{R}$. $\square$

Problem 11. (5 points) Make sure your name is on printed on the first page. $\square$