Mathematics 300

Quiz 17 Name: Key

You must show your work to get full credit.

1. Prove: If for any integers a, b, and c if abc is even, then at least one of a, b, or c is even. *Hint:* We have shown that the product of two odd numbers is odd. You can use this in the proof.

We prove the contrapositive: If a, b, c are all odd, then so is the product.

Assume a, b, c odd. Then the product ab is odd. But then $abc = (ab)c$ is the product of two odd numbers and thus odd.

done

2. Prove: For any integers n, if n^3 is even, then n is even.

We prove the contrapositive: If n is odd, then n^3 is odd.

Proof 1 Assume n odd. Then $n^3 = n \cdot n \cdot n$ is the product of three odd integers and thus n^3 is odd by theorem 1.

Proof 2 Assume n is odd. Then $n = 2q+1$ for some integer q. Then

$$
\begin{aligned}
n^3 &= (2q+1)^3 \\
&= (2q)^3 + 3(2q)^2 + 3(2q) + 1 \\
&= 8q^2 + 12q^2 + 6q + 1 \\
&= 2(4q^2 + 6q^2 + 3q) + 1 \\
&= 2m + 1
\end{aligned}
$$

where $m = 4q^2 + 6q^2 + 3q$ is an integer. Thus n^3 is odd.

1
1 1
1 2 1
1 3 3 1

3. Prove the number $\sqrt[3]{2}$ is irrational.

Towards a contradiction assume $\sqrt[3]{2}$ is rational. Let

$$\sqrt[3]{2} = \frac{p}{q}$$

where p, q are integers and this fraction is in lowest terms.

Cubing this equation gives

$$2 = \frac{p^3}{q^3}$$

so

(4) $$2q^3 = p^3$$

Thus $p^3 = 2m$ where $m = q^3$ is an integer. Thus p^3 is even. By problem 2 this gives that p is, say $p = 2s$. Use this in equation (4) to get

$$2q^3 = (2s)^3$$
$$2q^3 = 8s^3$$
$$q^3 = 4s^3 = 2(2s^2).$$

Thus q^3 is even (as $2s^2 \in \mathbb{Z}$). By problem 2 this implies q is even, say $q = 2t$ with $t \in \mathbb{Z}$. But then

$$\frac{p}{q} = \frac{2s}{2t}$$

contradicting that $\frac{p}{q}$ is in lowest terms

done

4. Prove: If α is irrational, then $\dfrac{\alpha+1}{\alpha-1}$ is irrational.

Toward a contradiction assume $\frac{\alpha+1}{\alpha-1}$ is rational.
Then

$$\frac{\alpha+1}{\alpha-1} = \frac{p}{q}$$

where p and q are integers. Then
(cross multiply)

$$(\alpha+1)q = (\alpha-1)p$$
$$q\alpha + q = p\alpha - p$$
$$q\alpha - p\alpha = -p-q$$
$$(q-p)\alpha = -p-q$$
$$\alpha = \frac{-p-q}{q-p} = \frac{p+q}{p-q}.$$
$$= \frac{m}{n}$$

where $m = p+q$, $n = p-q$ are integers. This implies α is rational, contradicting that it is irrational.