Mathematics 300

Quiz 21 Name: Key

You must show your work to get full credit.

1. Prove: For any integer n, the number $n^3 + 3n^2 - n + 3$ is divisible by 3.

We split the proof into three cases:
$n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$

Case 1 $n \equiv 0 \pmod 3$. Then

$$\begin{aligned} n^3+3n^2-n+3 &\equiv 0+0+0+3 \pmod 3 \\ &\equiv 3 \pmod 3 \\ &\equiv 0 \pmod 3 \end{aligned}$$

so $3 \mid (n^3+3n^2-n+3)$ in this case.

Case 2 $n \equiv 1 \pmod 3$. Then

using $3 \equiv 0 \pmod 3$

$$\begin{aligned} n^3+3n^2-n+3 &\equiv 1^3+0(1)^2-1+0 \pmod 3 \\ &\equiv 0 \pmod 3 \end{aligned}$$

so $3 \mid (n^3+3n^2-n+3)$ in this case also.

Case 3 $n \equiv 2 \pmod 3$. Then

$$\begin{aligned} n^3+3n^2-n+3 &\equiv 2^3+0-2+0 \pmod 3 \\ &\equiv 8-2 \pmod 3 \\ &\equiv 6 \pmod 3 \\ &\equiv 0 \pmod 3 \end{aligned}$$

so $3 \mid (n^3+3n^2-n+3)$ in this last case also

done

2. Prove or give a counterexample: For integers a and b, if $ab \equiv 0 \pmod{6}$, then $a \equiv 0 \pmod{6}$ or $b \equiv 0 \pmod{6}$.

This is false. The easiest counterexample is $a = 2 \not\equiv 0 \mod 6$, $b = 3 \not\equiv 0 \mod 6$ with $ab = 6 \equiv 0 \mod 6$.

3. Prove or give a counterexample: For integers a and b, if $ab \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$.

This is true. We prove the contrapositive:
If $a \not\equiv 0 \pmod 3$ and $b \not\equiv 0 \pmod 3$ then $ab \not\equiv 0 \pmod 3$
The proof of this splits into four cases.

<u>case 1</u> $a \equiv b \equiv 1 \pmod 3$. Then
$ab \equiv (1)(1) \equiv 1 \not\equiv 0 \pmod 3$

<u>case 2</u> $a \equiv 1 \pmod 3$ $b \equiv 2 \pmod 3$. Then
$ab \equiv (1)(2) \equiv 2 \not\equiv 0 \pmod 3$

<u>case 3</u> $a \equiv 2 \pmod 3$ $b \equiv 1 \pmod 3$. Then
$ab \equiv (2)(1) \equiv 2 \not\equiv 0 \pmod 3$

<u>case 4</u> $a \equiv b \equiv 2 \pmod 3$ Then
$ab \equiv (2)(2) \equiv 1 \not\equiv 0 \pmod 3$

<u>done.</u>

4. Prove or give a counterexample: If $4 \mid a^2$, then $4 \mid a$.

This is false. A counterexample is $a \equiv 2$. Then

$4 \mid a^2 \equiv 4$, but $4 \nmid a = 2$.

5. Prove or give a counterexample: If $3 \mid a^2$, then $3 \mid a$.

Proof 1 use problem 3 with $a=b$. Then

$3 \mid a^2 = ab$. so by problem 3 $3 \mid a$ or $3 \mid b = a$.

So $3 \mid a$.

Proof 2 we prove the contrapositive: If

$a \not\equiv 0 \pmod 3$. Then $a^2 \not\equiv 0 \pmod 3$.

The proof of this splits into two cases.

Case 1 $a \equiv 1 \pmod 3$

Then $a^2 \equiv 1^2 \equiv 1 \not\equiv 0 \pmod 3$

Case 2 $a \equiv 2 \pmod 3$.

Then $a^2 \equiv 2^2 \equiv 4 \equiv 1 \not\equiv 0 \pmod 3$ done

6. Prove $\sqrt{3}$ is irrational. That is show if r is a real number with $r^2 = 3$, then r is irrational.

Towards a contradiction assume r is rational. Then

$$r = \frac{a}{b}$$

with a, b integers and (Important point) we assume $\frac{a}{b}$ is in lowest terms.

Squaring gives

$$\frac{a^2}{b^2} = r^2 = 3$$

multiply by b^2

$$(*) \qquad a^2 = 3b^2$$

Then a^2 is 3 times the integer b^2 so $3 \mid a^2$. By Problem 5 this implies $3 \mid a$. Therefore

$$a = 3p \quad \text{for some } p \in \mathbb{Z}.$$

use this in $(*)$.

$$(3p)^2 = 3b^2$$
$$9p^2 = 3b^2$$
$$3p^2 = b^2$$

so b^2 is 3 times the integer p^2 and thus $3 \mid b^2$. Then (Problem 5 again) this implies $3 \mid b$. Thus

$$b = 3q \quad \text{for some } q \in \mathbb{Z}.$$

Thus

$$\frac{a}{b} = \frac{3p}{3q}$$

which contradicts that $\frac{a}{b}$ is in lowest terms. done