Mathematics 300

Quiz 23 Name: Key

You must show your work to get full credit.

1. Define the following:

(a) $a \equiv b \pmod{n}$. a, b, n are integers, $n \geq 1$ and $n \mid (a-b)$

(b) a is ***divisor*** of b. a, b are integers and $a = qb$ for some integer q.

(c) r is a ***rational*** number. $r = \frac{a}{b}$ where a, b are integers. (And $b \neq 0$, but we can just assume this as otherwise the fraction is not defined)

(d) r is ***irrational*** number.

r is not a rational number.

(e) The ***absolute value*** of the number a.

$$|a| = \begin{cases} a & a \geq 0 \\ -a & < 0 \end{cases}$$

2. State the following:

(a) The ***division algorithm***. If a, b are integers with $b > 0$, there are integers q (the quotient) and r (the remainder) with $a = qb + r$ and $0 \leq r < b$.

(b) The contrapositive of the statement: If n^3 is divisible by 3, then n is divisible by 3.

If n is not divisible by 3, then n^3 is not divisible by 3.

3. Prove or give a counterexample: $(x+y)^5 \equiv x^5 + y^5 \pmod{5}$.

This is true. <u>Proof</u>

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$\equiv x^5 + 0 + 0 + 0 + y^5 \pmod{5}$$
$$\equiv x^5 + y^5 \pmod{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

4. Compute the quotient and remainder when 7 is divided into -19.

$q =$ -3 $\qquad\qquad$ $r =$ 2

$-19 = -21 + 2 = (-3)(7) + 2$

so $q = -3$, $r = 2$

5. Prove: For any real numbers a and x, if $|x-1| = a$, then $x = a+1$ or $x = -a+1$.

There are two cases:

Case 1 $x - 1 \geq 0$. Then $|x-1| = x-1 = a$.

so $x = a+1$.

Case 2 $x - 1 < 0$. Then $|x-1| = -(x-1) = a$

so $x - 1 = -a$

$x = -a+1$.

So either $x = a+1$ or $x = -a+1$ done

6. Prove or give a counter example: If α is irrational, then $\dfrac{\alpha - 1}{\alpha + 1}$ is irrational.

Proof Towards a contradiction assume $\frac{\alpha-1}{\alpha+1}$ is rational. Then for some integers p, q.

$$\frac{\alpha - 1}{\alpha + 1} = \frac{p}{q}$$

$$q(\alpha - 1) = p(\alpha + 1)$$

$$q\alpha - q = p\alpha + p$$

$$q\alpha - p\alpha = p + q$$

$$(q - p)\alpha = p + q$$

$$\alpha = \frac{p+q}{q-p} = \frac{a}{b}$$

where $a = p+q$, $b = q-p$ are integers. This implies α is rational, contradicting our assumption that is irrational. done

7. Prove: For all integers n the number $n^3 + 5n$ is divisible by 3.

We do proof by cases. The three cases are $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$ $n \equiv 2 \pmod 3$

Case 1 $n \equiv 0 \pmod 3$

Then $n^3+5n \equiv 0+5\cdot 0 \equiv 0 \pmod 3$

Case 2 $n \equiv 1 \pmod 3$

Then $n^3+5n \equiv 1^3+5(1) \equiv 6 \equiv 0 \pmod 3$

Case 3 $n \equiv 2 \pmod 3$

Then $n^3+5n \equiv 2^3+5(2) \equiv 8+10 \equiv 18 \equiv 0 \pmod 3$.

so in all cases $n \equiv 0 \pmod 3$. But this implies $3 \mid n$.

8. We have shown that $\sqrt{3}$ is irrational. Use this to show that for any real number x that either x or $x - \sqrt{3}$ is irrational.

We do a proof by cases: x is irrational, or x is rational

Case 1 x is irrational. Then we are done.

Case 2 x is rational. Assume toward a contradiction that $x - \sqrt{3}$ is rational. Then for some integers a, b, c, d. That

$\frac{a}{b} = x$ and $\frac{c}{d} = x - \sqrt{3}$.

Solve for $\sqrt{3}$

$\sqrt{3} = x - \frac{c}{d} = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

imply that $\sqrt{3}$ is rational, contradicting that $\sqrt{3}$ is irrational

done

9. Prove or give a counterexample: If $3a \equiv 0 \pmod 6$, then $a \equiv 0 \pmod 6$.

This is false. If $a = 2$ then $a \not\equiv 0 \bmod 6$, but $3a = 6 \equiv 0 \pmod 6$

10. Prove or give a counterexample: If $3a \equiv 0 \pmod 5$, then $a \equiv 0 \pmod 5$.

This is true.

Proof 1 We prove the contrapositive: If $a \not\equiv 0 \pmod 5$ then $3a \not\equiv 0 \bmod 5$ by cases.

case 1 $a \equiv 1 \pmod 5$. Then $3a \equiv 3 \not\equiv 0 \bmod 5$

case 2 $a \equiv 2 \pmod 5$. Then $3a \equiv 6 \equiv 1 \not\equiv 0 \pmod 5$

case 3 $a \equiv 3 \pmod 5$. Then $3a \equiv 9 \equiv 4 \not\equiv 0 \pmod 5$

case 5 $a \equiv 4 \pmod 5$ Then $3a \equiv 12 \equiv 2 \not\equiv 0 \pmod 5$.

So in all the cases where $a \not\equiv 0 \pmod 5$ we have $3a \not\equiv 0 \pmod 5$ done

Proof 2 Assume $3a \equiv 0 \pmod 5$ Multiply by 2 to get $6a \equiv 0 \pmod 5$

But $6 \equiv 1 \pmod 5$ so this implies

$a \equiv 6a \equiv 0 \pmod 5$ done

11. Prove that if a and b have the same remainder when divided by n, then $a \equiv b \pmod n$.

If a and b have the same remainder when divided by n we have

$a = q_1 n + r$, $b = q_2 n + r$ with $0 \le r < n$.

Then

$$\begin{aligned} a - b &= (q_1 n + r) - (q_2 n + r) \\ &= q_1 n - q_2 n \\ &= (q_1 - q_2) n \\ &= q n \end{aligned}$$

where $q = q_1 - q_2 \in \mathbb{Z}$. Thus $n \mid (a-b)$.

This is the definition of $a \equiv b \pmod n$.