Mathematics 300

Quiz 26 Name: Key

You must show your work to get full credit.

Proposition. *If a, b, c, d, m are integers wtih $m \geq 1$ and*

$$a \equiv b \,(\mathrm{mod}\, m)$$
$$c \equiv c \,(\mathrm{mod}\, m)$$

then

$$ac \equiv bd \,(\mathrm{mod}\, m).$$

1. Use this proposition and mathematical induction to prove that for any integers a, b, and m with $m \geq 1$ that $a \equiv b \,(\mathrm{mod}\, m)$ implies $a^n \equiv b^n \,(\mathrm{mod}\, m)$ for all positive integers n.

Base case $n=1$: $a^1 = a$ and $b^1 = b$, and we are given $a \equiv b \mod m$. So the base case holds.

Induction step: Assume (induction hypothesis) that result is true for $n = k$, that is

$$a^k \equiv b^k \mod m.$$

We know $a \equiv b \mod m$.

Use the proposition with $c = a^k$, $d = b^k$ to conclude

$$a^{k+1} \equiv a a^k \equiv b b^k \equiv b^{k+1} \mod m$$

This finishes the induction step.

2. Use mathematical induction to prove that for any postive integer n

$$1+4+7+10+\cdots+(3n-2)=\frac{n(3n-1)}{2}.$$

<u>Base case</u> when $n=1$

$1+4+\ \ +(3n-2)=1$

and $\frac{n(3n-1)}{2}=\frac{1(3\cdot1-1)}{2}=\frac{1\cdot2}{2}=1.$

so $1+4+\ \ +(2n-2)=\frac{n(3n-1)}{2}$

holds in this case.

<u>Induction step</u> Assume (Induction hypothesis)

that result holds for $n=k$, i.e

(*) $1+4+\cdot+(2k-2)=\frac{k(3k-1)}{2}$

our goal is to show it holds for $n=k+1$, that is that

$1+4+\ \ +(3k-2)+(3\cdot(k+1)-2)=\frac{(k+1)(3(k+1)-1)}{2}$

which is the same as

(**) $1+4+\cdot\ \ +(3k-2)+(3k+1)=\frac{(k+1)(3k+2)}{2}$

Add $3(k+1)-2=3k+1$ to both sides of (*)

$1+4+\cdots+(3k-2)+(3(k+1)-2)=\frac{k(3k-1)}{2}+3k+1$

$=\frac{3k^2-k}{2}+\frac{2(3k+1)}{2}$

$=\frac{3k^2-k+6k+2}{2}$

$=\frac{3k^2+5k+2}{2}$

$=\frac{(k+1)(3k+2)}{3}$

so we have reached our goal of (**) which completes the induction.