Mathematics 300

Quiz 31 **Name:** Key

You must show your work to get full credit.

1. Use induction on a to prove the following special case of the ***division algorithm***. If a and b are positive integers, there there are integers q and r with

$$a = qb + r \qquad \text{with} \qquad 0 \leq r < b.$$

(Note we are not proving the uniqueness of q and r.)

Special case $b=1$. Then $a = a\cdot 1+0$, so we can just use $q=1$, $r=0$.

From now on assume $b \geq 2$, and we use induction on a.

Base case $a=1$. Then $a = 0\cdot b+1 = 1$. so it holds with $q=1$, $r=1$.

Induction step Assume true for $a=k$. That is $k = qb+r$ $0 \leq r < b$.

Case 1 $r < b-1$. Then

$$k+1 = qb+r+1 = qb+(r+1) = qb+r'$$

where $r' = r+1 < b-1+1 = b$ so we are done in this case

Case 2 $r = b-1$. Then

$$\begin{aligned} k+1 &= qb+r+1 \\ &= qb+(b-1)+1 \\ &= qb+b \\ &= (q+1)b \\ &= q'b+r' \end{aligned}$$

where $q' = q+1$, $r'=0$. so this case is done

2. Let $A = \{1, 2, 3, 4, 5, 6\}$ and B be the set of even integers between -3 and 9.

Write B as the list of its elements. $\{-2, 0, 2, 4, 6, 8\}$

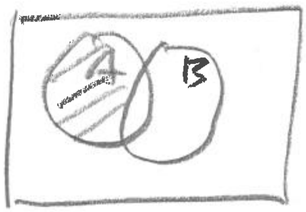
What is $A \cup B$? $\{-2, 0, 2, 3, 4, 5, 6, 8\}$

What is $A \cap B$? $\{2, 4, 6\}$

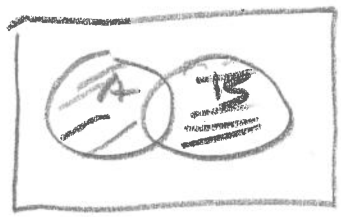
What is $A - B$ $\{1, 3, 5\}$

3. Draw the Venn diagrams for

(a) $A \cap B^c$



(b) $(A - B) \cup (B - A)$



(c) $(A \cup B) - (A \cap B)$.

(d) Are $(A - B) \cup (B - A)$ and $(A \cup B) - (A \cap B)$ equal? Why?

Yes, they have the same Venn diagram

4. Let a_n be defined recursively by

$$a_{n+1} = \frac{1}{2}a_n + 100, \qquad a_1 = 2$$

Prove that $a_n \leq 300$ for all n.

Base case: $a_1 = 2 < 300$ holds

Induction hypothesis: $a_k < 300$

Induction step: $a_{k+1} = \frac{1}{2}a_k + 100$

$< \frac{1}{2}(300) + 100$

$= 150 + 100$

$= 250$

< 300 done

5. Define the ***Fibonacci numbers*** by the recursion

$$f_{n+2} = f_{n+1} + f_n, \qquad f_1 = 1, \quad f_2 = 1$$

(a) Compute

$f_3 =$ 2 $\quad f_4 =$ 3 $\quad f_5 =$ 5 $\quad f_6 =$ 8 $\quad f_7 =$ 13

$1+1 \quad 1+2 \quad 2+3 \quad 3+5 \quad 5+13$

(b) Prove

$$\sum_{k=1}^{n} f_k = f_{k+2} - 1.$$

Base case: $n=1$. Then $\sum_{k=1}^{1} f_1 = f_1 = 1 = 2-1 = f_{1+2} - 1$ holds

Induction hypothesis: $f_1 + f_2 + \cdots + f_k = f_{k+2} - 1$

Add f_{k+1} to both sides of this

$f_1 + f_2 + \cdots + f_k + f_{k+1} = f_{k+1} + f_{k+2} - 1$

$= f_{k+3} - 1$

$= f_{(k+1)+2} - 1$

where we have used $f_{k+1} + f_{k+2} = f_{k+3}$.

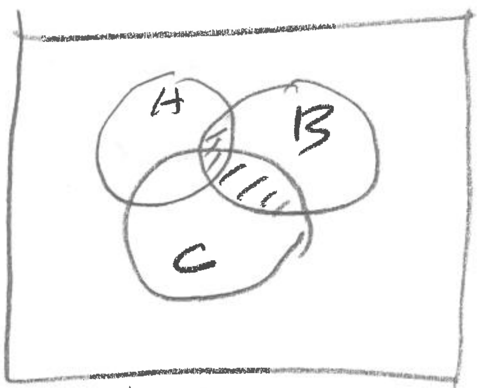
6. Give an example of two sets with $A \cap B \neq A \cup B$. The example should be explicitly given sets, not just a Venn diagram.

$A = \{1, 2\}$, $B = \{2, 3\}$

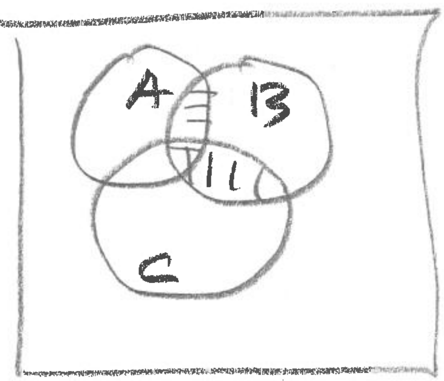
Then $A \cap B = \{2\}$, $A \cup B = \{1, 2, 3\}$

and these are not equal

7. Use Venn diagrams to show for sets A, C and B that $(A \cup C) \cap B = (A \cap B) \cup (C \cap B)$



$(A \cup C) \cap B$



$(A \cap B) \cup (C \cap B)$.

8. Let $A_1, A_2, \ldots, A_n, B$ be subsets of a set U. Use induction to show

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

Base case $n = 1$ $A_1 \cap B = A_1 \cap B$ is true.

Induction hypothesis

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B)$$

Induction step

$$(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) \cap B$$

$$= (\underbrace{(A_1 \cup \cdots \cup A_n)}_{A} \cup \underbrace{A_{n+1}}_{C}) \cap B \quad \leftarrow \text{In prob 7}$$

$$= ((A_1 \cup \cdots \cup A_n) \cap B) \cup (A_{n+1} \cap B)$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B) \cup (A_{n+1} \cap B).$$

$\leftarrow$ Ind. hyp.

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_{n+1} \cap B)$$

done