Mathematics 300 Homework, February 21, 2022.

Here is what is going to be on Wednesday's quiz.

Proposition 1. *For any integer n, if n^2 is even, then n is even.*

Proof. We will prove the contrapositive: If n is odd, then n^2 is odd.

Assume n is odd. Then $n = 2q + 1$ for some integer q. Therefore

$$n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1 = 2k + 1$$

where $k = 2q^2 + 2q$ is an integer. This shows n^2 is odd. □

Theorem 2. *The number $\sqrt{2}$ is irrational.*

Proof. Towards a contradiction assume $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers and $q \neq 0$. We also assume this fraction is in lowest terms. That is there is no integer $d > 1$ that divides both p and q. Take the equation $\sqrt{2} = p/q$ and square it to get

$$2 = \frac{p^2}{q^2}$$

and multiply this by q^2 to clear of fractions. The result is

$$(*) \qquad 2q^2 = p^2.$$

Therefore $p^2 = 2k$ with $k = q^2$ and thus p^2 is even. By the Proposition this implies p is even, say $p = 2m$ for some integer m. Use $p = 2m$ in equation $(*)$ to get

$$2q^2 = (2m^2) = 4m^2.$$

Divide by 2

$$q^2 = 2m^2 = 2\ell$$

where $\ell = m^2$ is an integer. Thus q^2 is even. Using the Proposition again we have that q is even, say $q = 2n$ for some integer. This implies our original fraction is

$$\frac{p}{q} = \frac{2m}{2n}$$

is not in lowest terms, a contradiction. □