Mathematics 300 Homework, March 2, 2022.

Our main new result today was the

Division Algorithm. *Let a and b be integers with $b \geq 1$. Then there are unique numbers integers q and r such that*

$$a = qb + r \qquad \text{and} \qquad 0 \leq r < b.$$

The integers q is the ***quotient*** *and r is the* ***remainder****.*

Problem 1. Find q and r in the following cases:

(a) $a = 55$ and $b = 2$.
(b) $a = 55$ and $b = 1$
(c) $a = 37$ and $b = 17$
(d) $a = -37$ and $b = 17$.

Note if $b = 2$, then $a = 2q + r$ where $0 \leq r < 2$. So the only possibilities for r are $r = 0$ and $r = 1$. When $r = 0$ this gives $a = 2q$ and when $r = 1$ we have $a = 2q + 1$ so this the two cases of a being even or odd.

When $b = 3$ we have

$$a = 3q + r \quad 0 \leq r < 3$$

so the only possibilities for r in this case are $r = 0$, $r = 1$, and $r = 2$. Here is an example of this using these three cases.

Proposition 1. *For any integer n the number $n(n+4)(n+8)$ is divisible by 3.*

Proof. We write

$$n = 3q + r \quad 0 \leq r < 3$$

so that $r = 0, 1, 2$. Split the proof into three cases

- $r = 0$. Then $n = 3q$ and we have

$$\begin{aligned} n(n+4)(n+8) &= (3q)(3q+4)(3q+8) \\ &= 3(q(3q+4)(3q+8)) \\ &= 3m \end{aligned}$$

  where $m = 3(q(3q+4)(3q+8))$ is an integers. So 3 divides $n(n+4)(n+8)$ in this case.
- $r = 1$. Then $n = 3q + 1$ and

$$\begin{aligned} n(n+4)(n+8) &= (3q+1)(3q+5)(3q+9) \\ &= 3((3q+1)(3q+5)(q+1)) \\ &= 3m \end{aligned}$$

  where $m = (3q+1)(3q+5)(q+1)$ is an integer. Therefore $n(n+4)(n+8)$ is even in this case.

- $r = 2$. Then $n = 31 + 2$ and

$$\begin{aligned} n(n+4)(n+8) &= (3q+2)(3q+6)((3q+10) \\ &= 3((3q+2)(q+2)(3q+10)) \\ &= 3m \end{aligned}$$

  where $m = (3q+2)(q+2)(3q+10)$ is an integer and therefore 3 divides $n(n+4)(n+8)$ in this case also.

□

Problem 2. Prove that the product of any four consecutive integers is divisible by 4. That is show that for any integers n the product $n(n+1)(n+2)(n+3)$ is divisible by 4. □

Problem 3. Show that there is no integer n such that n^2 has remainder 3 when divided by 4. *Hint:* Write $n = 4q+r$ where $r = 0, 1, 2, 3$ and show that in each of these cases the remainder when n^2 is divided by 4 the remainder is not 3. □

Problem 4. Show that for no integer n such that the remainder is 2 when n^2 is divided by 3. □