Mathematics 300 Homework, March 5, 2022.

Our most recent results are

Division Algorithm. *If a and b are integers with $b > 0$, then there are unique integers q (the quotient) and r (the remainder) with*

$$a = qb + r \qquad 0 \le r < b. \qquad \square$$

Theorem 1. *For any integers a and b and positive integer n $a \equiv b \pmod{n}$ if and only if a and b have the same remainder when divided by n.* $\square$

Theorem 2. *Let n be a positive integer. Then for all integers a, b, c*

(a) $a \equiv a \pmod{n}$ *for all integers a. (The **reflective property**).*
(b) *If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$ (The **symmetric property**).*
(c) *If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$ (The **transitive property**).*

Theorem 3. *Let n be a positive integer and a, b, c, d integers with*

$$a \equiv b \pmod{n} \qquad \text{and} \qquad c \equiv c \pmod{n}.$$

Then

$$a + c \equiv b + d \pmod{n} \qquad \text{and} \qquad ad \equiv bd \pmod{n}.$$

Also for any positive integer m

$$a^m \equiv b^m \pmod{n}.$$

Here is an example of how these results can simplify some of the proofs we have been doing.

Proposition. *For any integer n the number $n^3 - n$ is divisible by 3.*

Proof. The number $n^n - n$ is divisable by 3 if and only if $n^3 - n \equiv 3 \pmod{n}$. So we will show $n^3 - n \equiv 0 \pmod{3}$ for all n. There are three cases $n \equiv 0 \pmod{3}$, $n \equiv 1 \pmod{3}$ and $n \equiv 2 \pmod{3}$.

Case 1: $n \equiv 0 \pmod{3}$. Then

$$\begin{aligned} n^3 - n &\equiv 0^3 - 0 && \pmod{3} \\ &\equiv 0 && \pmod{3} \end{aligned}$$

Case 2: $n \equiv 1 \pmod{n}$. Then

$$\begin{aligned} n^3 - n &\equiv 1^3 - 1 && \pmod{3} \\ &\equiv 0 && \pmod{3} \end{aligned}$$

Case 3: $n \equiv 2 \pmod{3}$. Then

$$\begin{aligned} n^3 - n &\equiv 2^3 - 2 && \pmod{3} \\ &\equiv 6 && \pmod{3} \\ &\equiv 0 && \pmod{3} \end{aligned}$$

So in all cases $n^3 - n \equiv 0 \pmod{3}$ and therefore in all cases $3 \mid (n^3 - n)$. $\square$

Proposition. *For all integers n the number $n^5 - n$ is divisible by* 5.

Problem 1. Prove this by considering the cases $n \equiv 0 \pmod 5$, $n \equiv 1 \pmod 5$, $n \equiv 2 \pmod 5$, $n \equiv 3 \pmod 5$, $n \equiv 4 \pmod 5$.

Problem 2. Show that when n^2 is divided by 3 the remainder is never 2. *Hint:* This is the same as showing that for all n that $n^2 \not\equiv 2 \pmod 3$. Do this by looking at the cases $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$.

Problem 3. If an integer n is divided by 10, then the remainder is one of the numbers $r = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

(a) Let n be an positive integer and let d_0 be its last digit (the ones place). Explain why $n \equiv d_0 \pmod{10}$ For example if $n = 12{,}435$, then

$$n = (1{,}243)10 + 5$$

which shows n is of the form $10q + r$ where $r = 5$. Thus the remainder when n is divided by 10 is 5 and therefore $n \equiv 5 \pmod{10}$.

(b) The squares $\pmod{10}$ are

$$\begin{aligned}
0^2 &\equiv 0 \pmod{10}\\
1^2 &\equiv 1 \pmod{10}\\
2^2 &\equiv 4 \pmod{10}\\
3^2 &\equiv 9 \pmod{10}\\
4^2 &\equiv 6 \pmod{10}\\
5^2 &\equiv 5 \pmod{10}\\
6^2 &\equiv 6 \pmod{10}\\
7^2 &\equiv 9 \pmod{10}\\
8^2 &\equiv 4 \pmod{10}\\
9^2 &\equiv 1 \pmod{10}
\end{aligned}$$

Use this to explain why every perfect square ends in a $0, 1, 4, 6$ or 9.

(c) Explain why $n = 14{,}876{,}204{,}123$ is not a perfect square.

Problem 4. In the text in the problem set for Section 3.5 (Pages 153–157) do Problems 5,