Mathematics 300 Homework, March 30, 2022.

1. Summation notation.

This notation is very commonly used in mathematics, statistics, computer science, and most of the sciences and engineering to simplify writing sums.

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Thus

$$\sum_{k=0}^{5} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

There is nothing special about using k for the index:

$$\sum_{k=1}^{100} a_k = \sum_{j=1}^{100} a_j = \sum_{\alpha=1}^{100} a_\alpha = \sum_{\smiley=1}^{100} a_{\smiley} = \sum_{\frownie=1}^{100} a_{\frownie}.$$

Problem 1. Compute the following:

(a) $\displaystyle\sum_{k=1}^{4} k^2$

(b) $\displaystyle\sum_{j=3}^{6} (2j-1)$

(c) $\displaystyle\sum_{i=0}^{8} (-1)^8$

2. Geometric series.

A (finite) ***geometric series*** is a finite sum of the form

$$S = a + ar + ar^2 + \cdots + ar^n.$$

In summation notation this is

$$S = \sum_{k=0}^{n} ar^k.$$

Such sums occur naturally in many contexts and fortunately it is easy give a formula for their sum. We first look at the case of $n = 2$. Then

$$S = a + ar + ar^2.$$

Multiply this by r to get

$$rS = ar + ar^2 + ar^3.$$

Note that the sums for S and rS have the terms ar and ar^2 in common, which suggests subtracting to cancel these terms out:

$$\begin{aligned} S &= a + ar + ar^2 && \text{(Mutiply by } -r\text{)} \\ -rS &= \quad -ar - ar^3 - ar^3 && \text{(Mutiply by } -r\text{)} \\ S - rS &= a - ar^3 && \text{(Add these lines and cancel).} \end{aligned}$$

Therefore

$$(1-r)S = a - ar^3$$

which, when $r \neq 1$, we can solve for S to get

$$S = \frac{a - ar^3}{1-r}$$

For $n = 5$ the calculation looks like

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 \\ -rS &= \quad -ar - ar^2 - ar^3 - ar^4 - ar^5 - ar^6 && \text{(Mutiply by } -r\text{)} \\ S - rS &= a - ar^6 && \text{(Add these lines and cancel)} \end{aligned}$$

and therefore

$$(1-r)S = a - ar^6.$$

So when $r \neq 1$ we have

$$S = \frac{a - ar^6}{1-r}.$$

At this point you have likely guessed the general pattern:

Theorem 1. *Let a and r be real numbers with $r \neq 1$ and $n \geq 2$ and integer. Then the sum of the geometric series*

$$S = a + ar + ar^2 + \cdots + ar^n$$

is

$$S = \frac{a - ar^{n+1}}{1-r}.$$

Problem 2. Prove this. □

The way I find easiest to remember and apply this is to note that if the series $a + ar + ar^2 + \cdots + ar^n$ is continued that the next term would be ar^{n+1}. Therefore if we call the number r the ***ratio*** then

$$a + ar + ar^2 + \cdots + ar^n = \frac{\text{first} - \text{next term}}{1 - \text{ratio}}.$$

Here are some examples

$$x^2 + x^4 + x^6 + \cdots + x^{20} = \frac{\text{first} - \text{next term}}{1 - \text{ratio}} = \frac{x^2 - x^{22}}{1 - x^2}$$

holds when $x \neq \pm 1$.

Let

$$S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}.$$

Then

$$\begin{aligned} S &= \frac{1 - \text{next term}}{1 - \text{ratio}} \\ &= \frac{1-(-1/128)}{1-(-1/2)} = \frac{128+1}{128+64} = \frac{129}{192}. \end{aligned}$$

Let

$$\alpha = \overbrace{.333\cdots 3}^{n \text{ digits}}.$$

Then

$$\begin{aligned} \alpha &= 3(.1) + 3(.1)^2 + 3(.1)^3 + \cdots + 3(.1)^n \\ &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{3(.1) - 3(.1)^{n+1}}{1 - .1} \\ &= \frac{.3 - .3(.1)^n}{.3(3)} \\ &= \frac{1}{3} - \frac{1}{3(10)^n} \end{aligned}$$

There is anther natural way to find α:

$$\begin{aligned} 9\alpha = 10\alpha - \alpha &= (3.33\cdots 3) - (.333\cdots 3) \\ &= 3 - \underbrace{.000\cdots 3}_{10 \text{ decmal places}} \end{aligned}$$

Therefore

$$\alpha = \frac{3 - .000\cdots 3}{9} = \frac{1}{3} - \frac{.000\cdots 1}{3} = \frac{1}{3} - \frac{1}{3(10)^n}$$

Problem 3. Find the sums of the following geometric series

(a) $1 + 3 + 3^2 + 3^2 + \cdots + 3^n$

(b) $\displaystyle\sum_{k=2}^{n} \frac{5}{2^k}$

(c) $\displaystyle\sum_{k=0} 1{,}000(1.05)^k$

3. Some useful factoring formulas.

You recall that

$$x^2 - y^2 = (x-y)(x+y)$$

and may recall that

$$x^3 - y^3 = (x-y)(x^2+xy+y^2).$$

These generalize. To see how let us look at the right hand side of the last equation. If we multiple this out we get

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2) - y(x^2+xy+y^2)\\ &= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 \qquad \text{(most terms cancel)}\\ &= x^3 - y^3.\end{aligned}$$

Let us look at a similar product:

$$\begin{aligned}(x-y)(x^3+x^2y+xy^2+y^3) &= x(x^3+x^2y+xy^2+y^3) - y(x^3+x^2y+xy^2+y^3)\\ &= x^4 + x^3y + x^2y^2 + xy^3\\ &\qquad - x^3y - x^2y^2 - xy^3 - y^4\\ &= x^4 - y^4.\end{aligned}$$

And just to be sure we see the pattern let us look at the next case

$$\begin{aligned}(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) &= x(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &\qquad - y(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &= x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4\\ &\qquad - x^4y - x^3y^2 - x^2y^3 - xy^4 - y^5\\ &= x^5 - y^5.\end{aligned}$$

The pattern is now clear:

Theorem 2. *Let n be any positive integer and let x and y be any two real numbers. Then*

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}).$$

In summation notation this is

$$x^n - y^n = (x-y)\left(\sum_{k=0}^{n-1} x^{n-1-k}y^k\right) = (x-y)\left(\sum_{\substack{j+k=n-1\\ 0\le j,k\le n-1}} x^j y^k\right)$$

Problem 4. Prove this by multiplying out $(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$ and seeing that all but two terms cancel. □

Problem 5. The proof of Theorem 2 may remind you of the proof of Theorem 1 because both rely on a lot of cancellation. This is because there is a geometric series hidden in the proof of Theorem 2. Let use consider the case of $n = 5$ and set

$$S = x^4 + x^3y + x^2y^2 + xy^3 + y^4.$$

This can be written as

$$S = x^4 + x^4\left(\frac{y}{x}\right) + x^4\left(\frac{y}{x}\right)^2 + x^4\left(\frac{y}{x}\right)^3 + x^4\left(\frac{y}{x}\right)^4$$

which is a geometric series. Thus

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{x^4 - x^4\left(\frac{y}{x}\right)^5}{1 - \frac{y}{x}} \\ &= \frac{x^5 - y^5}{x - y}. \end{aligned}$$

Recalling the definition of S this is

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5 - y^5}{x - y}$$

which is equivalent to the $n = 5$ version of Theorem 2. Give a proof of general case of Theorem 2 using the method just given. □

4. Sums of arthritic series.

Anther sum that occurs naturally is

$$S = 1 + 2 + 3 + \cdots + n.$$

Let us compute this in some special cases. If

$$S = 1 + 2 + 3 + 4 + 5 + 6,$$

then also

$$S = 6 + 5 + 4 + 3 + 2 + 1.$$

We will add these together

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + 5 + 6 \\ S &= 6 + 5 + 4 + 3 + 2 + 1 \\ 2S &= (1 + 6) + (2 + 5) + (3 + 4) + (4 + 3) + (5 + 2) + (6 + 1) \\ &= 7 + 7 + 7 + 7 + 7 + 7 \qquad \text{(6 terms in the sum)} \\ &= 6 \cdot 7 \end{aligned}$$

Therefore

$$S = \frac{6 \cdot 7}{2} = 21.$$

This method works in general. If

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

then we can reverse the sum

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Adding these together gives

$$\begin{aligned} 2S &= (1+n) + (2+(n-1)) + (3+(n-2)) + \cdots + ((n-2)+3) + ((n-1)+2) + (n+1) \\ &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}_{n \text{ terms}} \\ &= n(n+1). \end{aligned}$$

Dividing by 2 gives

$$S = \frac{n(n+1)}{2}.$$

This gives a proof of

Theorem 3. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

□

This can be generalized a bit. In general a finite ***arithmetic series*** is a sum of the form

$$\begin{aligned} S &= a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d) \\ &= \sum_{k=0}^{n-1} (a+kd). \end{aligned} \tag{1}$$

This sum has n terms. The number d is the ***common difference*** (or just the ***difference***) of the series.

Problem 6. (This problem as much about leaning to use summation notation as it is about the result.) Use summation notation and the generalization of the argument given here for $n = 5$ to derive a formula for the sum of the series (1). When $n = 5$ we have

$$S = a + (a+d) + (a+2d) + (a+3d) + (a+4d) = \sum_{k=0}^{4} (a+kd)$$

Writing this sum in the reverse order

$$S = (a+4d) + (a+3d) + (a+2d) + (a+d) + a = \sum_{k=0}^{4} \big(a + (4-k)d\big)$$

Therefore

$$\begin{aligned}
2S &= S + S \\
&= \sum_{k=0}^{4}(a + kd) + \sum_{k=0}^{4}\big(a + (4-k)d\big) \\
&= \sum_{k=0}^{4}\big((a + kd + a + (4-k)d\big) \\
&= \sum_{k=0}^{4}(2a + 4d) \\
&= 5(2a + 4d).
\end{aligned}$$

Dividing by 2 then gives

$$S = 5(a + 2d) = 5a + 10d.$$

Here is a way to rewrite this to make is seem more intuitive.

$$\begin{aligned}
S = 5(a+2d) = 4\left(\frac{a + (a+4d)}{2}\right) = (\text{number of terms})\left(\frac{\text{first} + \text{last}}{2}\right) \\
= (\text{number of terms})\,(\text{average})
\end{aligned}$$

Thus the sum is the number of terms times the average of the first and last terms. Now you should do this argument in the case of general n. $\square$