Mathematics 300 Homework, April 1, 2022.

In class we talked about putting postage on post cards using stamps with just two denominations. To look at a example we did just briefly at the end of the hour assume we have 3¢ and 5¢ stamps and we want to know if we can put exactly n¢ on a post card. For small values we can just use brute force and make a table.

n	Possible?	How?
1	No	
2	No	
3	Yes	(1)3¢
4	No	
5	Yes	(1)5¢
6	Yes	(2)3¢
8	Yes	(1)3¢ + (1)5¢
9	Yes	(3)3¢
10	Yes	(2)5¢
11	Yes	(2)3¢ + (1)5¢
12	Yes	(4)3¢
13	Yes	(1)3¢ + (2)5¢

This makes is look like we can do any postage n with $n \geq 5$. The idea to do this is to assume (this is our induction hypothesis) exactly k¢ on a post card. Then we will either remove a 5¢ stamp and add in two 3¢ stamps or take out three 3¢ stamps and add in two 5¢ cent stamps for a total effect of adding 1¢. Note that since we may have to take out as much as three 3¢ stamps this will only work if $k \geq 9$.

Proposition 1. *Using 3¢ and 5¢ stamps we can put exactly n¢ postage on a post card for any $n \geq 5$*

Proof. **Base cases:** From the table we see it is possible to put on exactly n¢ for $n = 5, 6, 7, 8, 9$. So we only have to consider the cases were $n \geq 9$.

Induction step: Assume we can realize exactly k¢ and that $k \geq 9$. Our **induction goal** is to show that we can realize exactly $(k+1)$¢.

Case 1. There is a 5¢ stamp used to make the k¢ postage. Then remove this 5¢ stamp and replace it with two 3¢ stamps. The new postage is $k - 5 + 2(3) = k + 1$ and we have reached our goal.

Case 2. There are no 5¢ stamps used to get the postage of k¢. Then, as $k \geq 9$, there are at least three 3¢ stamps. Take out three 3¢ stamps and replace with two 5¢ stamps for a new total of $k - 3(3) + 2(5) = k + 1$. So we have reached our goal in this case also. □

Problem 1. Use only 2¢ and 5¢ stamps what postages are possible? Make a table and a conjecture and then prove your conjecture.

Problem 2. Use induction to prove

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

for all $n \geq 1$.

Problem 3. Be able to state the ***Division Algorithm***: If a, b are integers and $b > 0$ then there are unique integers q and r such that

$$a = qb + r \qquad \text{and} \qquad 0 \leq r < b.$$