Mathematics 300 Homework, April 5, 2022.

(4 + (24)(-2))

Problem 1. Be able to give the precise definition of $a \equiv b \pmod{n}$: If a, b, n are integers with $n0$ then

$$a \equiv b \pmod{n} \quad \text{if and only if} \quad n \mid (a - b).$$

Problem 2. In the text do Problem 17 on page 198.

We have just started to talk about ***recursion***, which, loosely speaking, is defining a function on the natural numbers by defining the value at n in terms of its values on previous integers.

Here is an example. Define a sequence by

$$a_0 = 0, \qquad a_{n+1} = 2 * a_n + 1.$$

We can then compute the first few values of a_n

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 2a_0 + 1 = 2 \cdot 0 + 1 = 1 \\
a_2 &= 2a_1 + 1 = 2 \cdot 1 + 1 = 3 \\
a_3 &= 2a_2 + 1 = 2 \cdot 3 + 1 = 7 \\
a_4 &= 2a_3 + 1 = 2 \cdot 7 + 1 = 15 \\
a_5 &= 2a_4 + 1 = 2 \cdots 15 + 1 = 31 \\
a_6 &= 2a_5 + 1 = 2 \cdot 31 + 1 = 63
\end{aligned}$$

and we can keep going as long as we want and this is an example of a recursive definition. It would be time consuming to compute a_{100} be this method by hand. (But, as I am sure some of you have already realized, it is very easy to get a computer to do this.)

In this example we can guess a formula by noting

$$\begin{aligned}
a_0 &= 2^0 - 1 \\
a_1 &= 1 = 2^1 - 1 \\
a_2 &= 3 = 2^2 - 1 \\
a_3 &= 7 = 2^3 = 1 \\
a_4 &= 15 = 2^4 - 1 \\
a_5 &= 31 = 2^5 - 1 \\
a_6 &= 63 = 2^6 - 1
\end{aligned}$$

So a good guess is that $a_n = 2^n - 1$.

Proposition 1. *Let a_n be defined recursively by*

$$a_0 = 0, \qquad a_{n+1} = 2 * a_n + 1.$$

Then

$$a_n = 2^n - 1.$$

Proof. We use induction on n.

Base case. $n = 0$. Then $a_0 = 0 = 2^0 - 1$ so this case works.

Induction hypothesis: $a_k = 2^k - 1$.

Induction goal: $a_{k+1} = 2^{k+1} - 1$.

Assuming the induction hypothesis, that is $a_k = 2^k - 1$ and using the formula for a_{k+1} we have

$$a_{k+1} = 2a_k - 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$

which closes the induction and completes the proof. □

Problem 3. Let a_n be defined by

$$a_0 = 2, \qquad a_{n+1} = -3a_n + 4.$$

(a) Use $a_{n+1} = -3a_n + 4$ and $a_0 = 2$ to compute a_1, a_2, a_3, and a_4.
(b) Use induction to prove $a_n = (-3)^n + 4$. □

A little more complicated are two step recursions. As an example let

$$f(0) = 2, \quad f(1) = 5, \qquad f(n+2) = 5f(n+1) - 6f(n).$$

Again we can compute as many values as we want step by step:

$$\begin{aligned}
f(0) &= 2 \\
f(1) &= 5 \\
f(2) &= 5f(1) - 6f(0) = 5(5) - 6(2) = 13 \\
f(3) &= 5f(2) - 6f(1) = 5(13) - 6(5) = 35 \\
f(4) &= 5f(3) - 6f(2) = 5(35) - 6(13) = 97 \\
f(5) &= 5f(4) - 6f(3) = 5(97) - 6(35) = 275
\end{aligned}$$

The pattern here is not so clear, but there is a formula.

Proposition 2. *Let $f(n)$ satisfy*

$$f(1) = 2, \quad f(2) = 6, \qquad f(n+2) = 5f(n+1) - 6f(n)$$

Then

$$f(n) = 2^n + 3^n$$

Proof. We use induction on n.

Base cases: $n = 0$ and $n = 2$.

$$\begin{aligned}
f(0) &= 2 = 2^0 + 3^0 \\
f(1) &= 5 = 2^1 + 3^1
\end{aligned}$$

both hold.

Induction hypothesis: $f(j) = 2^j + 3^k$ for $1 \le j \le k$.

Induction goal: $f(k+1) = 2^{k+1} + 3^{k+1}$

By the induction hypothesis we have

$$f(k-1) = 2^{k-1} + 3^{k-1}, \qquad f(k) = 2^k + 3^k$$

Therefore

$$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1}) \\ &= 5(2^k) - 6(2^{k-1}) + 5(3^k - 6(3^{k-1}) \\ &= (5(2) - 6)2^{k-1} + (5(3) - 6))3^{k-1} \\ &= 4(2^{k-1}) + 9(3^{k-1}) \\ &= 2^2(2^{k-1}) + 3^2(3^{k-1}) \\ &= 2^{k+1} + 3^{k+1} \end{aligned}$$

which completes the induction and the proof. □

Problem 4. In the problems on pages 208–209 do 6, 8, 12.