Mathematics 300 Test 1 Name: Key

Show your work to get credit.

The problems are worth 10 points each.

1. Define the following:

(a) The integer n is *even*. $n = 2q$ for some integer q.

(b) The integer n is *odd*. $n = 2q+1$ for some integer q

(c) The integer a ***divides*** the integer b (in symbols $a \mid b$).

$a \neq 0$ and $b = qa$ for some integer q.

(d) For integers a, b, n the a is ***congruent to*** b ***modulo*** n. (in symbols $a \equiv b \pmod{n}$.)

$n \mid (a-b)$

2. Prove or give a counterexample: If n is even, then $(n^3+11) \equiv 3 \pmod{8}$.

Proof Assume n is even. Then

$n = 2q$ for some integer.

Thus

$$
\begin{aligned}
(n^3+11) - 3 &= (2q)^3 + 8 \\
&= 8q^3 + 8 \\
&= 8(q^2+1) \\
&= 8k
\end{aligned}
$$

where $k = q^2+1$ is an integer by closure properties. This shows

$$8 \mid ((n^3+11) - 3)$$

which is the definition of

$$n^3 + 11 \equiv 3 \pmod{8}$$

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1. Define the following:
   (a) The integer n is ***even***.

$n = 2q$, for some integer q.

   (b) The integer n is ***odd***.

$n = 2q+1$ for some integer q.

   (c) The integer a ***divides*** the integer b (in symbols $a \mid b$).

$a \neq 0$ and $b = q(a)$ for some integer q.

   (d) For integers a, b, n the a is ***congruent to*** b ***modulo*** n. (in symbols $a \equiv b \pmod{n}$.)

$a \equiv b \pmod{n}$ if $n \mid (a-b)$ and $n \neq 0$

2. Prove or give a counterexample: If n is even, then $(n^3 + 11) \equiv 3 \pmod{8}$.

Assume n is even

Then $n = 2q$, for some integer q

$n^3 + 11 - 3 = (2q)^3 + 11 - 3$ substitution $2q$ in for n

$= 8q^3 + 8$

$= 8(q^3 + 1)$

$= 8(q')$ where $q' = q^3 + 1$ is an integer by closure properties

Then $8 \mid ((n^3 + 11) - 3)$ by the def of divides

$\therefore$ $(n^3 + 11) \equiv 3 \pmod{8}$ by the def of congruent to modulo.

100

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38
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98

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3. (a) Make truth tables for $P \to (P \to Q)$ and $\neg P \wedge Q$.

For $P \Rightarrow (P \Rightarrow Q)$

P	Q	$P \Rightarrow Q$	$P \Rightarrow (P \Rightarrow Q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

For $\neg P \wedge Q$

P	Q	$\neg P$	$\neg P \wedge Q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

not same

(b) Are $P \to (P \to Q)$ and $\neg P \wedge Q$ logically equivalent? Explain your answer.

No they are not logically equivalent as their truth tables are not the same

4. Let P be the statement "If the number a is positive, then the equation $f(x) = a$ has a solution".

(a) What is the converse of this statement?

If the equation $f(x)=a$ has a solution, then a is positive.

(b) What is the contrapositive of this statement?

If the equation $f(x)=a$ does not have a solution, then a is not positive

(c) What is the negation of this statement?

The number a is positive and the equation $f(x)=a$ has no solution.

5. What is the negation of the statement "Every even number is the sum of two prime numbers"?

There exists an even number that is not the sum of two primes

6. (a) Write $\{x \in \mathbb{Z} : (2x+1)(x-4) = 0\}$ in roster notation. $\{4\}$

$x = 4, -\frac{1}{2}$

$-\frac{1}{2} \notin \mathbb{Z}$

(b) $\{x \in \mathbb{Q} : (2x+1)(x-4) = 0\}$ in roster notation. $\{4, -\frac{1}{2}\}$

$x = 4, -\frac{1}{2}$

(c) Write $\{-2, -1, 0, 1, 2, \ldots, 101, 102\}$ in set builder notation. $\{n \in \mathbb{Z} : -2 \leq n \leq 102\}$

3. (a) Make truth tables for $P \to (P \to Q)$ and $\neg P \wedge Q$.

P	Q	$P \to Q$	$P \to (P \to Q)$	$\neg P$	$\neg P \wedge Q$
T	T	T	T	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	F

(b) Are $P \to (P \to Q)$ and $\neg P \wedge Q$ logically equivalent? Explain your answer.

No, they are not logically equivalent because they do not have the same truth table values

4. Let P be the statement "If the number a is positive, then the equation $f(x) = a$ has a solution".

(a) What is the converse of this statement?

If the equation $f(x) = a$ has a solution, then the number a is positive.

(b) What is the contrapositive of this statement?

If the equation $f(x) = a$ does not have a solution, then the number a is not positive.

(c) What is the negation of this statement?

The number a is positive and the equation $f(x) = a$ does not have a solution

5. What is the negation of the statement "Every even number is the sum of two prime numbers"?

There exists an even number that is not the sum of two prime numbers.

6. (a) Write $\{x \in \mathbb{Z} : (2x+1)(x-4) = 0\}$ in roster notation.

$-\frac{1}{2}$ 4

$\{4\}$

(b) $\{x \in \mathbb{Q} : (2x+1)(x-4) = 0\}$ in roster notation.

$-\frac{1}{2}$ 4

$\{-\frac{1}{2}, 4\}$

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(c) Write $\{-2, -1, 0, 1, 2, \ldots, 101, 102\}$ in set builder notation.

$\{x \in \mathbb{Z} : -2 \leq x \leq 102\}$

7. Prove the transitive law for congruence. That is prove: If $r \equiv s \pmod{m}$ and $s \equiv t \pmod{m}$, then $r \equiv t \pmod{m}$.

Assume $r \equiv s \pmod{m}$ and $s \equiv t \pmod{m}$

Then

$m \mid (r-s)$ and $m \mid (s-t)$.

Thus there are integers p, q with

$r - s = pm$

$s - t = qm$

Add these equations

$r - \cancel{s} + \cancel{s} - t = pm + qm$

$r - t = (p+q)m = km$

where $k = p + q$ is an integer by closure properties. This is the definition of

$m \mid (r-t)$

which in turn is the definition of $r \equiv t \pmod{m}$ done

8. Prove: If $a \equiv b \pmod{n}$ and $x \equiv y \pmod{n}$, then $a - x \equiv b - y \pmod{n}$.

Assume $a \equiv b \pmod{n}$ and $x \equiv y \pmod{n}$

Then $n \mid (a-b)$ and $n \mid (x-y)$

So there are integers p, q with

$a - b = pn$

$x - y = qn$.

Substract these equations

$(a-b) - (x-y) = pn - qn$

by algebra $(a-x) - (b-y) = (p-q)n = kn$

where $k = p - q$ is an integer by closure properties. Thus

$n \mid ((a-x) - (b-y))$

which is the definition of

$(a-x) \equiv (b-y) \pmod{n}$ done

7. Prove the transitive law for congruence. That is prove: If $r \equiv s \pmod{m}$ and $s \equiv t \pmod{m}$, then $r \equiv t \pmod{m}$.

Assume $r \equiv s \pmod{m}$ and $s \equiv t \pmod{m}$, then $m \mid (r-s)$ and $m \mid (s-t)$ meaning... there are some integers p and q such that $r-s = mp$ and $s-t = mq$. Then, $r = mp+s$ and $t = s-mq$. By substitution,

$$\begin{aligned} r-t &= (mp+s)-(s-mq) \\ &= mp+mq \\ &= m(p+q) \\ &= mr, \end{aligned}$$

where $r = p+q$ is an integer by closure properties. Therefore, $m \mid (r-t)$, which means by definition $r \equiv t \pmod{m}$

good

8. Prove: If $a \equiv b \pmod{n}$ and $x \equiv y \pmod{n}$, then $a - x \equiv b - y \pmod{n}$.

Assume $a \equiv b \pmod{n}$ and $x \equiv y \pmod{n}$, then $n \mid (a-b)$ and $n \mid (x-y)$. So, there are integers p and q such that $(a-b) = pn$ and $(x-y) = qn$. By algebra,

$$a = pn+b$$
$$x = qn+y, \text{ so}$$

$$\begin{aligned} a-x &= pn+b-(qn+y) \\ &= pn-qn+b-y \\ &= n(p-q)+(b-y) \end{aligned}$$

and

$$\begin{aligned} (a-x)-(b-y) &= n(p-q) \\ &= nr, \end{aligned}$$

where r is an integer by closure properties.

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Therefore, $n \mid (a-x)-(b-y)$, so by definition $(a-x) \equiv (b-y) \pmod{n}$

9. Prove or give a counterexample: If $x \equiv y \pmod{n}$, then $x^2 \equiv y^2 \pmod{n}$.

Proof Assume $x \equiv y \pmod{n}$. Then

$n \mid (x-y)$.

so there is an integer q with

$x - y = nq$.

By algebra

$x = y + nq$.

so $x^2 = (y+nq)^2$

$= y^2 + 2nyq + n^2q^2$

$x^2 - y^2 = 2nyq + n^2q$

$= n(2yq + nq)$

$= nk$

where $k = (2yq + nq) \in \mathbb{Z}$ by closure propoerties of $\mathbb{Z}$. Thus

$n \mid (x^2 - y^2)$. This implies

$x^2 \equiv y^2 \pmod{n}$ done

10. Find all Pythagorean triples of the form m, $m+3$, $m+6$.

The triples are: 9, 12, 15

For $a = m$, $b = m+3$, $c = m+6$ to be a Pythagorean triple we need $a^2 + b^2 = c^2$. That is

$m^2 + (m+3)^2 = (m+6)^2$

$\cancel{m^2} + m^2 + 6m + 9 = \cancel{m^2} + 12m + 36$

$m^2 - 6m - 27 = 0$

$(m-9)(m+3) = 0$

so $m = 9, -3$.

But m must be positive so only $m = 9$ works. Thus The triple is

$a = m = 9$

$b = m+3 = 12$

$c = m+6 = 15$

9. Prove or give a counterexample: If $x \equiv y \pmod{n}$, then $x^2 \equiv y^2 \pmod{n}$.

Proof

we assume $x \equiv y \pmod{n}$. By definition, $n \mid x-y$. So by definition, there is an integer k such that $nk = x-y$. Using substitution,

$$x^2 = (nk+y)^2 = n^2k^2 + 2nky + y^2.$$

Using algebra,

$$x^2 - y^2 = n^2k^2 + 2nky$$
$$= n(nk^2 + 2ky)$$

where p is the integer $nk^2 + 2ky$.

So $x^2 - y^2 = np$. Therefore $n \mid x^2 - y^2$ so by definition, $x^2 \equiv y^2 \pmod{n}$

10. Find all Pythagorean triples of the form m, $m+3$, $m+6$.

The triples are: 9, 12, 15

$$m^2 + (m+3)^2 = (m+6)^2$$
$$m^2 + m^2 + 6m + 9 = m^2 + 12m + 36$$
$$2m^2 + 6m + 9 = m^2 + 12m + 36$$
$$m^2 - 6m - 27 = 0$$
$$(m-9)(m+3) = 0$$
$$m = 9, -3$$

-3 is NOT a natural number

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