Mathematics 300 Test 2 Name: Key

Show your work to get credit.

The problems are worth 10 points each.

1. (20 points) Define the following: $n \mid (a-b)$ where a, b, n are integers and $n \neq 0$

(a) $a \equiv b \pmod{n}$.

(b) a is a ***divisor*** of b. written $a \mid b$. $\Leftrightarrow$ $a \neq 0$ and there is an integer q so that $b = aq$.

(c) r is a ***rational number***. $r = \frac{a}{b}$ where a, b are integers and $b \neq 0$

(d) The ***absolute value*** of the number b.

$$|b| = \begin{cases} b & b \geq 0 \\ -b & b < 0 \end{cases}$$

2. (10 points) (a) Give a precise statement of the ***division algorithm***.

For any integers a, b with $b > 0$, there are unique integers q and r so that $a = qb + r$ and $0 \leq r < b$

(b) What are the quotient and remainder when -42 is divided by 9.

$q =$ -5 $r =$ 3

$$-42 = -45 + 3$$
$$= (-5)(9) + 3$$
so $q = -5$, $r = 3$

Mathematics 300 Test 2

Show your work to get credit.

The problems are worth 10 points

1. (20 points) Define the following:

(a) $a \equiv b \pmod{n}$.

for integers a, b, n, n divides the quantity a−b. So there is some integer k where $nk = a-b$

(b) a is a ***divisor*** of b.

for integers a, b, there is some integer q so that $aq = b$.

(c) r is a ***rational number***.

r can be represented as $\frac{a}{b}$ where a, b are integers, $b \neq 0$, and a and b are in lowest terms.

(d) The ***absolute value*** of the number b.

The absolute value of b is b when $b \geq 0$ and the absolute value of b is −b when $b < 0$.

2. (10 points) (a) Give a precise statement of the ***division algorithm***.

For integers a and b where $b \geq 1$, there are unique integers q and r where $0 \leq r < b$ so that $a = qb + r$.

(b) What are the quotient and remainder when −42 is divided by 9.

$q =$ −5 $\qquad r =$ 3

30
20
20
20
10
100

30

3. (10 points) Use definitions to prove: If $a \equiv 0 \pmod{n}$, then $n \mid a$.

By definition $a \equiv 0 \pmod{n}$ if and only if

$n \mid a - 0 = a$. done

4. (10 points) Prove: If β is irrational, then $\dfrac{2-\beta}{1+\beta}$ is irrational.

We prove the contrapositive: If $\frac{2-\beta}{1+\beta}$ is rational, then β is rational.

Assume $\frac{2-\beta}{1+\beta}$ is rational. Then

$$\frac{2-\beta}{1+\beta} = \frac{a}{b} \quad \text{with } a, b \in \mathbb{Z},\ b \neq 0.$$

$$b(2-\beta) = a(1+\beta)$$
$$2b - b\beta = a + a\beta$$
$$-a\beta - b\beta = a - 2b$$
$$a\beta + b\beta = -(a-2b) = 2b - a$$
$$(a+b)\beta = 2b - a$$
$$\beta = \frac{2b-a}{a+b} = \frac{p}{q}$$

where $p = 2b - a$, $q = a + b$ are integers.

Thus β is rational. done

3. (10 points) Use definitions to prove: If $a \equiv 0 \pmod{n}$, then $n \mid a$.

we can assume $a \equiv 0 \pmod{n}$, then by definition of congruence, we know $n \mid (a-0)$

this means $a = nq$ for some integer q

$a = n \cdot k$ where $k = q$ is an integer by closure ~~properties~~.

By the last equation, we see that if $a \equiv 0 \pmod{n}$, then $n \mid a$.

4. (10 points) Prove: If β is irrational, them $\frac{2-\beta}{1+\beta}$ is irrational.

<u>Proof</u> Towards a contradiction, we assume $\frac{2-\beta}{1+\beta} = \frac{p}{q}$ for some integers p and q, and $q \neq 0$

By algebra:

$$(2-\beta)q = (1+\beta)p$$

$$2q - \beta q = p + \beta p$$

$$2q - p = \beta q + \beta p$$

$$\beta(q+p) = 2q - p$$

$$\beta = \frac{2q-p}{q+p} = \frac{m}{n}$$

where $m = 2q - p$ and $n = q + p$ are integers by closure ~~properties~~ and $n \neq 0$

From the last equation, we ~~see~~ β is rational which is a contradiction.

20

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

5. (10 points) Prove or give a counterexample: For any integers a and b

$$(a+b)^4 \equiv a^4 + b^4 \pmod 2.$$

This is True.

By Pascal's triangle we have

$$(*) \quad (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

But $4 \equiv 0 \pmod 2$, $6 \equiv 0 \pmod 2$.

So reducing $(*)$ mod 2 gives

$$(a+b)^4 \equiv a^4 + 0 + 0 + 0 + b^4 \pmod 2$$

$$\equiv a^4 + b^4 \pmod 2$$

6. (10 points) Prove or give a counterexample: For all integers a, b, n with $n \geq 1$, if $ab \equiv 0 \pmod n$, then $a \equiv 0 \pmod n$ or $b \equiv 0 \pmod n$.

This is False. An easy counter example is

$a = 2$, $b = 3$, $n = 6$. Then

$a = 2 \not\equiv 0$ mod 6

$b = 3 \not\equiv 0$ mod 6

but $ab = 6 \equiv 0$ mod 6.

5. (10 points) Prove or give a counterexample: For any integers a and b

$$(a+b)^4 \equiv a^4 + b^4 \pmod{2}.$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\equiv a^4 + 0 + 0 + 0 + b^4 \pmod{2}$

$\equiv a^4 + b^4 \pmod{2}$

Done.

6. (10 points) Prove or give a counterexample: For all integers a, b, n with $n \geq 1$, if $ab \equiv 0 \pmod{n}$, then $a \equiv 0 \pmod{n}$ or $b \equiv 0 \pmod{n}$.

8 6

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Counterexample: let $a = 8$, $b = 3$, $n = 6$. Then

$(ab) = (8)(3) = 24 \equiv 0 \pmod{6}$,

but $8 \equiv 2 \not\equiv 0 \pmod{6}$ and

$3 \equiv 3 \not\equiv 0 \pmod{6}$.

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7. (10 points) Find all solutions to $|x+3| = 9$. $x =$ 6, -12

We consider two cases

Case 1 $x+3 \geq 0$. Then $|x+3| = x+3 = 9$

so $x = 9-3 = \underline{6}$

Case 2 $x+3 < 0$ Then $|x+3| = -(x+3) = 9$

so $x+3 = -9$

$x = -9-3 = -\underline{12}$

As a check:

If $x=6$, $|x+3| = |6+3| = 9$.

If $x=-12$ $|x+3| = |-12+3| = |-9| = 9$.

8. (10 points) Prove: If a^3 is divisible by 3, then a is divisible by 3.

We prove the contrapositive: If a is not divisible by 3, then a^3 is not divisible by 3.

Assume a is not divisible by 3. Then $a \not\equiv 0 \pmod 3$ which leaves the two cases $a \equiv 1 \bmod 3$ or $a \equiv 2 \pmod 3$).

Case 1 $a \equiv 1 \pmod{3}$.)

Then $a^3 \equiv 1^3 \equiv 1 \not\equiv 0 \pmod 3$).

so a^2 is not divisible by 3

Case 2 $a \equiv 2 \pmod 3$). Then

$a^3 \equiv 2^3 \equiv 8 \equiv 2 \not\equiv 0 \pmod 3$),

so a^3 is not divisible by 3 in this case either.

done

7. (10 points) Find all solutions to $|x+3| = 9$.

$x =$ 6, -12

$|x+3| = 9$

$x+3 \geq 0$ then $x+3 = 9$

$x = 6$

$x+3 < 0$ then $-(x+3) = 9$

$-x - 3 = 9$

$-x = 12$

$x = -12$

$x = 6, -12$

8. (10 points) Prove: If a^3 is divisible by 3, then a is divisible by 3.

Proof: We will prove the contrapositive:

If a is not divisible by 3, then a^3 is not divisible by 3

There are two cases: $a \equiv 1 \pmod 3$, $a \equiv 2 \pmod 3$

good

Case 1: $a \equiv 1 \pmod 3$

Then $a^3 \equiv (1)^3 \pmod 3$

$a^3 \equiv 1 \pmod 3$

Case 2: $a \equiv 2 \pmod 3$

$a^3 \equiv (2)^3 \pmod 3$

$a^3 \equiv 8 \pmod 3$

$a^3 \equiv 2 \pmod 3$

Thus, in all cases if a is not divisible by 3, then a^3 is not divisible by 3

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9. (10 points) Use Problem 8 to prove $\sqrt[3]{3}$ is irrational.

Towards a contradiction assume $\sqrt[3]{3}$ is rational. Then

$$\sqrt[3]{3} = \frac{a}{b} \quad \text{with } a, b \text{ integers}$$

and we assume this is in lowest terms. Cubing gives

$$3 = \frac{a^3}{b^3}$$

$$(*) \qquad 3b^3 = a^3.$$

Thus a^3 is divisible by 3 (as it is 3 times the integer b^3). By problem 8 a is divisible by 3. So $a = 3p$ for some integer p. Use this in $(*)$

$$3b^3 = (3a)^3 = 27a^3$$

$$b^3 = 9a^3 = 3(3a^3)$$

so b^3 is divisible by 3 as it is 3 times the integer $3a^3$. Thus (problem 8 again) 3 divides b. Thus $b = 3q$ for some integer q. Then

$$\frac{a}{b} = \frac{3p}{3q}$$

contradicting that $\frac{a}{b}$ is in lowest terms. done

9. (10 points) Use Problem 8 to prove $\sqrt[3]{3}$ is irrational.

Proof.

Towards a proof of contradiction assume $\sqrt[3]{3}$ is rational, such that

$$\sqrt[3]{3} = \frac{p}{q} \text{ with } p, q \in \mathbb{Z} \text{ and } q \neq 0$$

and it must be in lowest terms.

Then cube both sides

$$3 = \frac{p^3}{q^3}$$

$$(*) \qquad 3q^3 = p^3$$

Thus p^3 is 3 times q^3, which means that $3 \mid p^3$, and based on knowledge from Problem 8, $3 \mid p$.

Therefore $p = 3m$ $m \in \mathbb{Z}$, which can then be used in $(*)$,

$$3q^3 = (3m)^3$$

$$3q^3 = 27m^3$$

$$q^3 = 9m^3$$

This shows that q^3 is 9 times m^3, and as 9 is divisible by 3 (it has no remainder), this means that $3 \mid q^3$, and by Problem 8 $3 \mid q$ $\quad q \equiv 0 \pmod 3$

Therefore $q = 3n$, $n \in \mathbb{Z}$.

Then, using this information

$$\frac{p}{q} = \frac{3m}{3n}$$

which is a contradiction, as it is not in lowest terms. Which proves that $\sqrt[3]{3}$ is not rational.

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