Mathematics 300 Test 3 Name: Key

Show your work to get credit.

The problems are worth 10 points each.

1. (10 points) (a) Give a precise statement of the ***division algorithm***.

For all integers a, b with $b > 0$ there are unique integers q, r so that

$a = qb + r$ and $0 \le r < b$

(b) What are the quotient and remainder when -17 is divided by 3?

The quotient is -6 The remainder is 1

$-17 = -18 + 1 = -6(3) + 1$
$= q(3) + 1$

2. (10 points) Give the definitions of the following:

(a) a ***divides*** b (in symbols $a \mid b$).

$a \neq 0$ and $b = qa$ for some $q \in \mathbb{Z}$

(b) $a \equiv b \pmod{n}$.

$n \mid (a - b)$

(c) The ***absolute value*** of the real number x.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

3. (5 points) Find the following sum:

$$S = 3 - 3(2) + 3(2)^2 - 3(2)^3 + 3(2)^4 - 3(2)^5 + 3(2)^6$$

$S =$ $2^7 + 1 = 129$

This is a geometric series with ratio $= -2$.

Thus

$$S = \frac{\text{first} - \text{next}}{1 - \text{ratio}}$$

$$= \frac{3 - (-3(2)^7)}{1 - (-2)}$$

$$= \frac{3 + 3(2)^7}{3} = 1 + 2^7$$

100/100

Mathematics 300 Test 3 Name:

Show your work to get credit.

The problems are worth 10 points each.

1. (10 points) (a) Give a precise statement of the *division algorithm*.

If a, b are integers and $b>0$, there are unique integers q, r such that:

$a = qb + r, \quad 0 \le r < b$

(b) What are the quotient and remainder when -17 is divided by 3?

The quotient is -6 The remainder is 1

$-6(3)+1$

$-18+1=-17$

Well done

2. (10 points) Give the definitions of the following:

(a) a ***divides*** b (in symbols $a \mid b$).

If a, b are integers, $a \neq 0$, a divides b if for some integer q, $qa = b$

(b) $a \equiv b \pmod n$.

If a, b, n are integers, $n \neq 0$, $a \equiv b \pmod n$ if and only if $n \mid (a-b)$

(c) The ***absolute value*** of the real number x.

$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

25
25
15
10
10
100

3. (5 points) Find the following sum:

$$S = 3 - 3(2) + 3(2)^2 - 3(2)^3 + 3(2)^4 - 3(2)^5 + 3(2)^6$$

$S = \dfrac{3 + 3(2)^7}{1+2} = 2^7 + 1 = 129$

-2 $+2$

$\dfrac{\text{first} - \text{next}}{1 - \text{ratio}}$ $\dfrac{1 + 3(2)^7}{1+2}$

129

$3 + 384$

25

4. (5 points) Find the sum of the following 30 terms:

$$A = 40 + 42 + 44 + \cdots + 96 + 98.$$

$A =$ 2070

This is an arithmetic series so

$A = (\#\text{ of terms})(\text{Average})$

$= 30\left(\frac{40+98}{2}\right)$

$= 30(20+49)$

$= 30(69) = 2070$

5. (10 points) (a) Let n be an integer. Give the precise definition of n being ***odd***.

$n = 2q+1$ for some integer q.

(b) Prove or give a counterexample: Every odd integer is the sum of three odd integers.

Let n be odd, say $n = 2q+1$ for some integer q. Then

$n = (2q+1) + (1) + (-1)$

and $2q+1$, 1, and -1 are all odd, so n is the sum of 3 odd numbers

6. (10 points) We have shown that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$. Use this fact and induction to prove: If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ for all positive integers n.

Base case: $n=1$. $a^1 = a$, $b^1 = b$ so $a^1 \equiv b^1 \pmod{m}$ as $a \equiv b \pmod{m}$

Induction hypothesis: $a^k \equiv b^k \pmod{m}$

Induction goal: $a^{k+1} \equiv b^{k+1} \pmod{m}$

Induction step: use $c = a^k$, $d = b^k$ in what we are given. Then the induction hypothesis is that $c \equiv d \pmod{m}$. Then

$ac \equiv bd \pmod{m}$

i.e. $aa^k \equiv b\,b^k \pmod{m}$

$a^{k+1} \equiv b^{k+1} \pmod{m}$. done

4. (5 points) Find the sum of the following 30 terms:

$$A = 40 + 42 + 44 + \cdots + 96 + 98.$$

$S =$ 2070

4 — — — — —
5 — — — — —
6
7
8
9

$30\left(\frac{40+98}{2}\right)$

$30\left(\frac{138}{2}\right) = 30(69)$

5. (10 points) (a) Let n be an integer. Give the precise definition of n being **odd**.

There is some integer k such that $n = 2k+1$

(b) Prove or give a counterexample: Every odd integer is the sum of three odd integers.

Proof
we assume n is odd. so $n = 2k+1$ where k is an integer. 1 is odd, and by definition, $2k$ is even. So n is the sum of an odd and even number. If we subtract 1 from $2k$, we get $2k-1$. So $2k$ is the sum of $2k-1$ and 1. Again, 1 is odd. By definition $2k-1$ is odd. So, $n = (2k-1) + 1 + 1$ which are three odd integers.

good

6. (10 points) We have shown that if $a \equiv b\,(\text{mod } m)$ and $c \equiv d\,(\text{mod } m)$, then $ac \equiv bd\,(\text{mod } m)$. Use this fact and induction to prove: If $a \equiv b\,(\text{mod } m)$, then $a^n \equiv b^n\,(\text{mod } m)$ for all positive integers n.

Proof by Induction. we assume $a \equiv b(\text{mod } m)$

Base case: $n=1$. $a^1 \equiv b^1 \,(\text{mod } m)$ is $a \equiv b \,(\text{mod } m)$ which is true because it is our assumption.

Induction Hypothesis: Assume $a^k \equiv b^k \,(\text{mod } m)$

$a^{k+1} = a^k a$ and $b^{k+1} = b^k b$. Using the fact that $ac \equiv bd \,(\text{mod } m)$ we can say $a^k a \equiv b^k b \,(\text{mod } m)$ since we have that $a^k \equiv b^k \,(\text{mod } m)$. Thus,

$a^{k+1} \equiv b^{k+1} \,(\text{mod } m)$

25

7. (15 points) Chicken nuggets come in boxes with either 3 nuggets or 4 nuggets. Prove that for any $n \geq 8$ that is possible to buy boxes so that the total number of nuggets is n.

We use $n=8$ as base case. Then we can get 8 nuggets by getting 2 packages of size 4.

Induction hypothesis $k \geq 8$ and we can get packs totaling k nuggets

Induction goal: we can get packs totaling $k+1$ nuggets

Induction step: If our packs totaling k has at least one pack of size 3, swap it for a pack of size 4 to get

$$k-3+4=k+1$$ nuggets

so we are done in this case.

The only case left is if all the packs we have are of size 4. Then (as $k \geq 8$) there are at least two of them. So swap two size 4 packs for three size 3 packs. Then our new collection has

$$k-2(4)+3(3)=k-8+9=k+1$$

nuggets. done

7. (15 points) Chicken nuggets come in boxes with either 3 nuggets or 4 nuggets. Prove that for any $n \geq 8$ that is possible to buy boxes so that the total number of nuggets is n.

n	Possible	
1	No	
2	No	
3	Yes	$\boxed{3}$
4	Yes	$\boxed{4}$
5	No	
6	Yes	$\boxed{3}\boxed{3}$
7	Yes	$\boxed{3}\boxed{4}$
8	Yes	$\boxed{4}\boxed{4}$

holds at $n=8$

For $n \geq 8$ there are 2 cases to look at, there is at least 1 3 nuggets box, or there are no nuggets boxes ✓

Inductive step: assume holds for $n = 8$ to k, so it is possible to make k total nuggets from 3 or 4 boxes.

Case 1: there is at least 1 3 nugget box, take away the 3 nugget box and add a 4 nugget box.

$$k - 3(1) + 4(1) = k+1 \text{ nuggets}$$

Case 2: there are no 3 nugget boxes so take away 2 4 boxes, and add 3 3 nugget boxes

$$k - 4(2) + 3(3) = k+1$$

∴ for $n \geq 8$ it is possible to make n number of nuggets with only 3 nugget and 4 nugget boxes.

Nice Job

15

8. (15 points) Define a function recursively by

$$f(0) = 5, \qquad f(n+1) = 3f(n) + 4$$

(a) Compute the following:

$f(1) =$ 19 $\qquad f(2) =$ 61 $\qquad f(3) =$ 187

$f(1) = 3f(0) + 4 = 3(5) + 4 = 15 + 4 = 19$

$f(2) = 3f(1) + 4 = 3(19) + 4 = 57 + 4 = 61$

$f(3) = 3f(2) + 4 = 3(61) + 4 = 183 + 4 = 187$

(b) Prove $f(n) = 7(3)^n - 2$ for all positive integers.

Base case $n = 0$. Then $7(3)^0 - 2 = 7(1) - 2 = 5$ so this works.

Induction hypothesis: $f(k) = 7(3)^k - 2$

Induction Goal: $f(k+1) = 7(3)^{k+1} - 2$

Induction step:

$$\begin{aligned} f(k+1) &= 3f(k) + 4 \\ &= 3(7(3)^k - 2) + 4 \qquad \text{(Sub in } f(k) = 7(3^k) - 2\text{)} \\ &= 7(3)^{k+1} - 6 + 4 \\ &= 7(3)^{k+1} - 2 \end{aligned}$$

and we have reached the induction goal and so have finished the proof.

8. (15 points) Define a function recursively by

$$f(0) = 5, \qquad f(n+1) = 3f(n) + 4$$

(a) Compute the following:

$f(1) =$ 19 $\qquad f(2) =$ 61 $\qquad f(3) =$ 187

(b) Prove $f(n) = 7(3)^n - 2$ for all positive integers.

Proof By Induction

Base Case: $n = 1$, $f(1) = 7(3)^1 - 2$ HOLDS

$= 21 - 2 = 19$

Induction Hyp: $f(k) = 7(3)^k - 2$

Induction Goal: $f(k+1) = 7(3)^{k+1} - 2$

Induction Step: $f(k+1) = 3(f(k)) + 4$

$= 3(7(3)^k - 2) + 4$ Good

$= 7(3)^{k+1} - 6 + 4$

$= 7(3)^{k+1} - 2$

DONE

15

9. (10 points) Let $a_1, a_2, a_3, \ldots$ be a sequence of numbers with

$$a_1 = 17, \qquad \text{and} \qquad a_{n+1} = \frac{1}{2}a_n + 42.$$

Prove $a_n < 84$ for all positive integers n.

Base case $a_1 = 17 < 84$ holds

Induction hypothesis: $a_k < 84$

Induction goal: $a_{k+1} < 84$

Induction step:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2} a_k + 42 \\ &< \tfrac{1}{2}(84) + 42 \quad (\text{as } a_k < 84 \\ &= 42 + 42 \\ &= 84 \end{aligned}$$

done.

9. (10 points) Let $a_1, a_2, a_3, \ldots$ be a sequence of numbers with

$$a_1 = 17, \qquad \text{and} \qquad a_{n+1} = \frac{1}{2}a_n + 42.$$

Prove $a_n < 84$ for all positive integers n.

Prove by induction

Base case $n=1$ $\quad a_1 = 17 \quad 17 < 84$ holds

Induction Step

Hypothesis $\quad a_k < 84$

Goal: Prove $a_{k+1} < 84$

$a_{k+1} = \frac{1}{2}a_k + 42$

Induction

$a_{k+1} < \frac{1}{2}(84) + 42$ by hypothesis

$a_{k+1} < 42 + 42$

$a_{k+1} < 84$

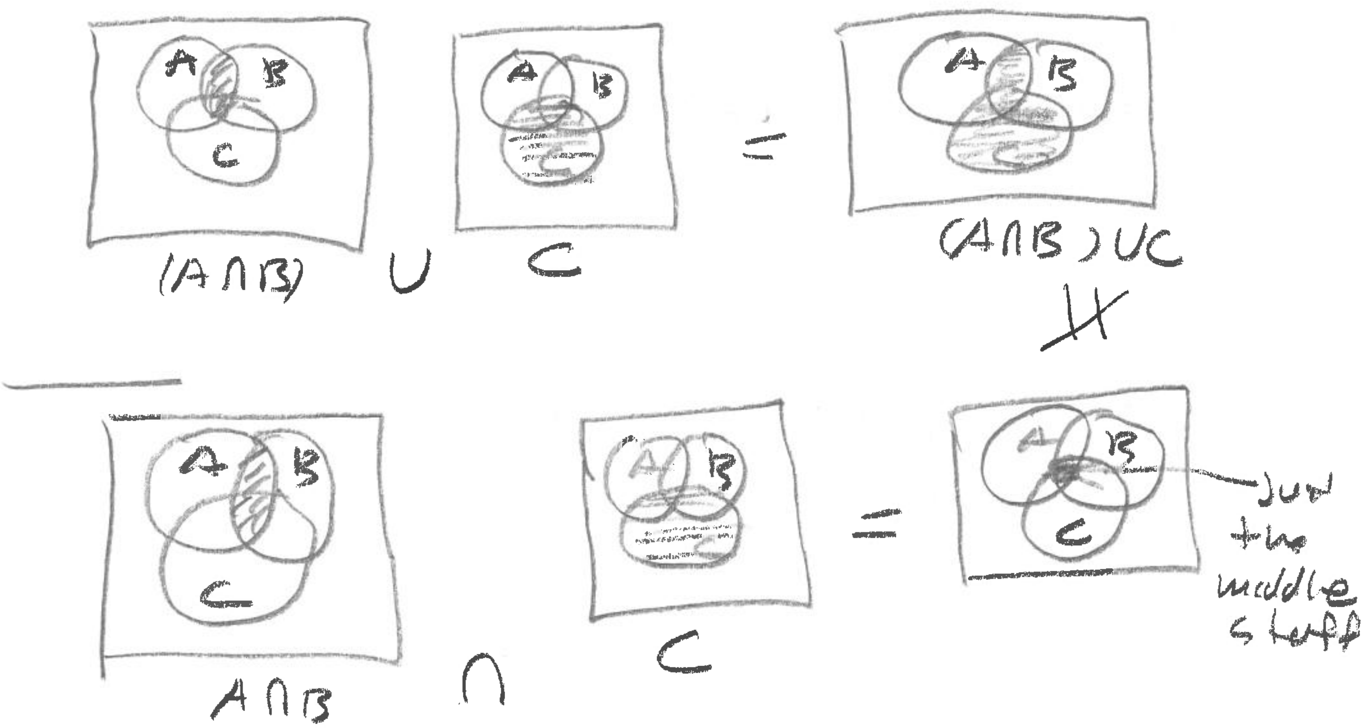
done

10

10. (10 points) True or False: For any sets A, B, and C

$$(A \cap B) \cup C = (A \cap B) \cap C.$$

Use Venn diagrams to explain your answer.



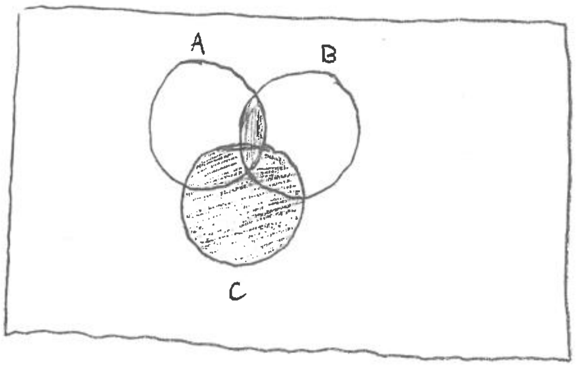
So it is false as the Venn diagrams are different.

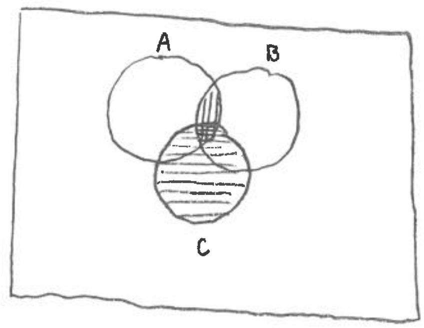
10. (10 points) True or False: For any sets A, B, and C

$$(A \cap B) \cup C = (A \cap B) \cap C.$$

Use Venn diagrams to explain your answer.

$(A \cap B) \cup C$



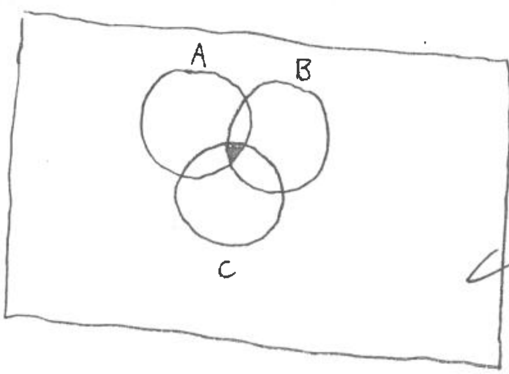


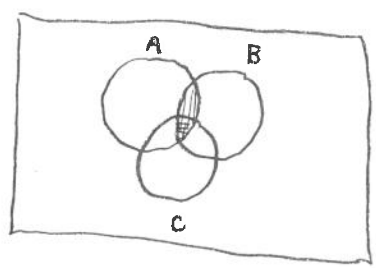
||| = $A \cap B$

≡ = $\cup\, C$

every thing shaded = $(A \cap B) \cup C$

$(A \cap B) \cap C$





||| = $A \cap B$

≡ = $\cap\, C$

井 = $(A \cap B) \cap C$

The proposition is false, because the venn diagrams for $(A \cap B) \cup C$ and $(A \cap B) \cap C$ are not the same.

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