Contents

1. Riemann Integration

We start with some definitions.

Definition 1. Let $[a, b]$ be a closed bounded interval. Then a ***partition*** of $[a, b]$ is a list of points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We denote it by $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$. We also use the notation

$$\Delta x_j = x_j - x_{j-1}.$$

(See Figure 1.) □

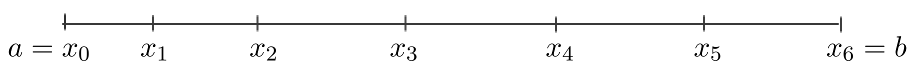

Figure 1. A partition of the interval $[a, b]$ into $n = 6$ pieces. The j-th interval $[x_{j-1}, x_j]$ has length $\Delta x_j = x_j - x_{j-1}$.

Definition 2. The function φ is a ***step function*** on $[a, b]$ so that there is a partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ and φ is constant on each open interval (x_{j-1}, x_j) for $j = 1, 2, \dots, n$. We denote the set of all step functions on $[a, b]$ by $\mathcal{S}[a, b]$. □

Proposition 3. *The set $\mathcal{S}[a, b]$ is a vector space. That if if $\varphi_1, \varphi_2 \in \mathcal{S}[a, b]$ and $c_1, c_2 \in \mathcal{S}[a, b]$, then $c_1\varphi_1 + c_2\varphi_2 \in \mathcal{S}[a, b]$.*

Problem 1. Prove this. □

Proposition 4. *If f is a bounded function on the closed bounded interval $[a, b]$ then f is integrable if and only if all $\varepsilon > 0$ there are step functions $\varphi, \psi \in \mathcal{S}[a, b]$ such that*

$$\varphi \le f \le \psi$$

and

$$\int_a^b (\psi - \varphi)\, dx < \varepsilon.$$

Problem 2. Prove this. *Hint:* We outlined the proof in class. □

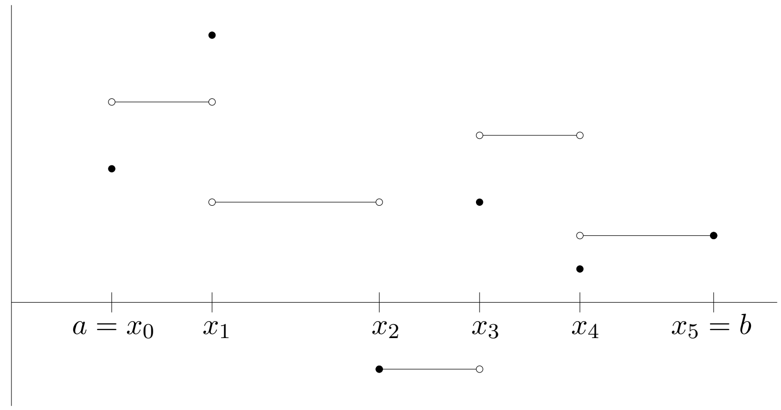

FIGURE 2. A step function, φ, for the interval $[a,b]$ partitioned into five subintervals. By definition φ is constant on each of the open intervals (x_{j-1}, x_j) for $j = 1, 2, 3, 4, 5$. No assumption is made about the values at the points x_j.

Definition 5. Let $\varphi \in \mathcal{S}[a,b]$ and let $\mathcal{P} = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition such that φ has the constant value c_j on the open interval (x_{j-1}, x_j). Then the ***integral*** of φ on the interval $[a,b]$

$$\int_a^b \varphi(x)\,dx = \sum_{j=1}^n c_j(x_j - x_{j-1}).$$ □

Setting $\Delta x_j = x_j - x_{j-1}$ (the length of the interval (x_{j-1}, x_j)) this can also be written as

$$\int_a^b \varphi(x)\,dx = \sum_{j=1}^n c_j \Delta x.$$

When a all the c_j's are positive, this is just the area under the graph of φ computed by adding up the area rectangles. In the general case (where the c_j's can be negative) this is the area where the area below the graph x-axis counted as negative.

Proposition 6. *The integral is linear on* $\mathcal{S}[a,b]$. *That is if* $\varphi_1, \varphi_2 \in \mathcal{S}[a,b]$, *and* $c_1, c_2 \in \mathbb{R}$, *then*

$$\int_a^b (c_1\varphi_1 + c_2\varphi_2)\,dx = c_1 \int_a^b \varphi_1(x)\,dx + c_2 \int_a^b \varphi_2(x)\,dx.$$ □

Problem 3. Prove this. □

Definition 7. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then define the ***upper integral*** of f to be

$$\overline{\int_a^b} f(x)\,dx = \inf \left\{ \int_a^b \psi(x)\,dx : \psi \in \mathcal{S}[a,b] \text{ and } f(x) \le \psi(x) \text{ all } x \in [a,b] \right\}.$$

Likewise the ***lower integral*** of f is

$$\underline{\int_a^b} f(x)\,dx = \inf\left\{ \int_a^b \psi(x)\,dx : \varphi \in \mathcal{S}[a,b] \text{ and } f(x) \geq \varphi(x) \text{ all } x \in [a,b] \right\}.$$

Definition 8. The bounded function $f\colon [a,b] \to \mathbb{R}$ is ***Riemann integrable*** if and only if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

In this case the ***integral*** of f is the common value of the upper and lower integral

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

We denote the set of all Riemann integrable functions on $[a,b]$ by $\mathcal{R}[a,b]$.

The following gives a method for showing that a function is Riemann integrable without having to compute the upper and lower integrals.

Theorem 9. *The bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for all $\varepsilon > 0$ there is are step functions $\varphi, \psi \in \mathcal{S}[a,b]$ with*

$$\varphi \leq f \leq \psi$$

on $[a,b]$ and

$$\int_a^b (\psi(x) - \varphi(x))\,dx < \varepsilon.$$

Proof. Done in class. □

Theorem 10. *The set $\mathcal{R}[a,b]$ is a vector space and the integral is linear on $\mathcal{R}[a,b]$.*

Proof. This was also done in class. □

If f is a monotone increasing function on $[a,b]$ and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ define two step functions by $\varphi_{f,\mathcal{P}}(b) = f(b)$,

$$\varphi_{f,\mathcal{P}}(x) = f(x_{j-1}) \qquad \text{for} \qquad x \in [x_{j-1}, x_j)$$

and $\psi_{f,\mathcal{P}}(b) = f(b)$

$$\psi_{f,\mathcal{P}} = f(x_j) \qquad \text{for} \qquad x \in [x_{j-1}, x_j).$$

See Figure 3

Proposition 11. *If f is monotone increasing on $[a,b]$ then for any partition, $\mathcal{P}$, of $[a,b]$, with the notation above,*

$$\varphi_{f,\mathcal{P}} \leq f \leq \psi_{f,\mathcal{P}}$$

on $[a,b]$.

Problem 4. Prove this. □

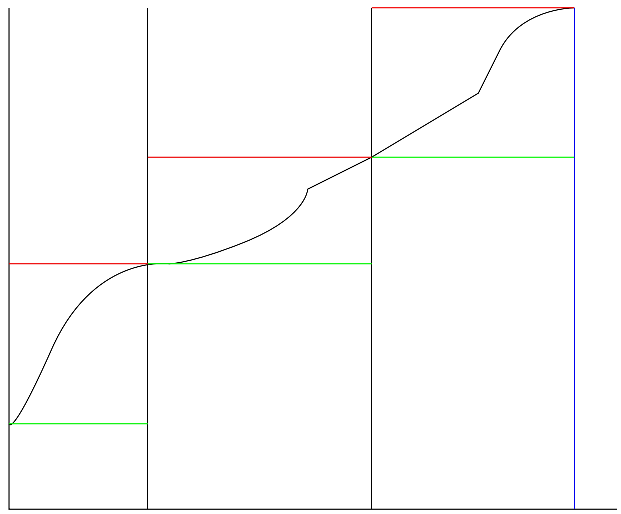

FIGURE 3. A monotone increasing function on $[a,b]$ and a partition, $\mathcal{P}$, with $n = 3$ showing the lower step function $\varphi_{f,\mathcal{P}}$ (in green) and the upper step function $\psi_{f,\mathcal{P}}$ (in red).

Definition 12. Given a positive integer n and a closed bounded interval $[a,b]$ the ***uniform partition*** of $[a,b]$ into n sub-intervals is the partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ with

$$x_j = a + j\left(\frac{b-a}{n}\right)$$

for $j = 0, 1, \dots, n$. Note in this case all the lengths, Δx_j of the sub-intervals $[x_{j-1}, x_j]$ have the same value $\Delta x = \Delta x_j = (b-a)/n$. □

Now let us consider the monotone increasing function f on the interval $[a,b]$ with the uniform partition, $\mathcal{P}$, of $[a,b]$ with $n = 4$. Then $\Delta x = \Delta x_j = (b-a)/4$ and $\varphi_{f,\mathcal{P}} \le f \le \psi_{f,\mathcal{P}}$. Also

$$\int_a^b \varphi_{f,\mathcal{P}}(x)\,dx = \big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\big)\Delta x$$

and

$$\int_a^b \psi_{f,\mathcal{P}}(x)\,dx = \big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\big)\Delta x.$$

Thus

$$\int_a^b \left(\psi_{f,\mathcal{P}}(x) - \psi_{f,\mathcal{P}}(x)\right)\,dx = \left(f(x_4) - f(x_0)\right)\Delta x = \left(f(b) - f(a)\right)\Delta x$$

There is nothing special about $n = 4$ in this:

Problem 5. Show that if f is monotone increasing on $[a,b]$, n is a positive integer and $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ is the uniform partition of $[a,b]$ into n

sub-intervals, then, with the notation above,

$$\int_a^b (\psi_{f,\mathcal{P}}(x) - \varphi_{f,\mathcal{P}}(x))\, dx = (f(b) - f(a))\,\Delta x = \frac{(f(b)-f(a))(b-a)}{n}. \quad \square$$

Theorem 13. *If f is a monotone function on the closed bounded interval $[a, b]$, then f is integrable on $[a, b]$.*

Problem 6. Prove this. *Hint:* With out loss of generality assume f is monotone increasing (if f is monotone decreasing replace f by $-f$). Let $\varepsilon > 0$ and let n be a positive integer such that

$$\frac{(f(b)-f(a))(b-a)}{n} < \varepsilon$$

and use Proposition 4 and the last problem. □

Theorem 14. *Let f be a continuous function on $[a, b]$. Then f is integrable on $[a, b]$.*

Proof. Let $\varepsilon > 0$. As f is continuous on the closed bounded set $[a, b]$ it is uniformly continuous on $[a, b]$. Thus there is an $\delta > 0$ such that for $x, y \in [a, b]$.

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Let n be a positive integer such that

$$\frac{b-a}{n} = \Delta x < \delta$$

and let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be the uniform partition of $[a, b]$ into n sub-intervals. Set

$$\begin{aligned} m_j &= \inf\{f(x) : x \in [x_{j-1}, x_j]\} = \min\{f(x) : x \in [x_{j-1}, x_j]\}, \\ M_j &= \sup\{f(x) : x \in [x_{j-1}, x_j]\} = \max\{f(x) : x \in [x_{j-1}, x_j]\} \end{aligned}$$

where the infimum is achieved as a minimum and the supremum is achieved as a maximum because continuous functions on closed bounded sets achieve their maximums and minimums. Define step functions φ and ψ on $[a, b]$ $\varphi(b) = \psi(b) = f(b)$ and

$$\begin{aligned} \varphi(x) &= m_j \quad \text{for} \quad x_{j-1} \le x < x_j \\ \psi(x) &= M_j \quad \text{for} \quad x_{j-1} \le x < x_j. \end{aligned}$$

Then

$$\varphi \le f \le \psi$$

and

$$\int_a^b (\varphi(x) - \psi(x))\, dx = \sum_{j=1}^n (M_j - m_j)\left(\frac{b-a}{n}\right).$$

As f is continuous on the closed bounded interval $[x_{j-1}, x_j]$, f achieves its maximum and minimum on this interval. Thus there are $\alpha_j, \beta_j \in [x_{j-1}, x_j]$

with $f(\alpha_j) = m_j$ and $f(\beta_j) = M_j$. But then $|\alpha_j - \beta_j| \le \Delta x < \delta$ and therefore

$$M_j - m_j = |f(\beta_j) - f(\alpha_j)| < \frac{\varepsilon}{b-a}.$$

Thus

$$\int_a^b (\varphi(x) - \psi(x))\,dx = \sum_{j=1}^n (M_j - m_j)\left(\frac{b-a}{n}\right) < \sum_{j=1}^n \frac{\varepsilon}{b-a}\left(\frac{b-a}{n}\right) = \varepsilon$$

and the result now follows from Proposition 4. □

Lemma 15. *Let* $\alpha, \beta \in \mathbb{R}$, *then*

$$|\max\{\alpha, 0\} - \max\{\beta, 0\}| \le |\alpha - \beta|.$$

Problem 7. Prove this by splitting it into the four cases *(i)* $\alpha, \beta \ge 0$, *(ii)* $\alpha \ge 0, \beta < 0$, *(iii)* $\alpha < 0, \beta \ge 0$, and *(iv)* $\alpha, \beta < 0$. This is not to be handed in. □

Proposition 16. *If* $f \in \mathcal{R}[a,b]$ *then so is* $g = \max\{f, 0\}$.

Proof. Let $\varepsilon > 0$ Let φ and ψ be step functions on $[a,b]$ such that $\varphi \le f \le \psi$ and $\int_a^b (\psi - \varphi)\,dx < \varepsilon$. Then

$$\varphi_0 = \max\{0, \varphi\}, \qquad \psi_0 = \max\{0, \psi\}$$

are step functions, $\varphi_0 \le \max\{f, 0\} \le \psi_0$ and $0 \le \psi_0 - \varphi_0 \le \psi - \varphi$. Thus, using Lemma 15,

$$\int_a^b (\psi_0 - \varphi_0)\,dx \le \int_a^b (\psi - \varphi)\,dx < \varepsilon$$

and so $\max\{f, 0\}$ is integrable by Proposition 4. □

This implies a good deal more because of the following elementary result.

Lemma 17. *For real numbers* a, b *the following hold*

$$\begin{aligned} \min\{a, 0\} &= -\max\{-a, 0\}, \\ |a| &= \max\{a, 0\} + \max\{-a, 0\}, \\ \max\{a, b\} &= a + \max\{0, b - a\}, \\ \min\{a, b\} &= a + \min\{0, b - a\}. \end{aligned}$$

Proof. Left to reader (and you don't have to turn these in). We did enough of this type of thing last term that I believe you can do it. □

Proposition 18. *If* f *and* g *are integrable on* $[a,b]$ *then so are* $|f|$, $\min\{f, g\}$ *and* $\max\{f, g\}$.

Proof. This follows easily from Proposition 16 and Lemma 17. □

Lemma 19. *If* f *is integrable on* $[a,b]$ *then so is* f^2.

Problem 8. Prove this. *Hint:* As $f^2 = |f|^2$ and $|f|$ is also integrable by replacing f by $|f|$ we can assume $f \geq 0$. As f is integrable it is bounded, say $0 \leq f \leq B$ on $[a, b]$. Also as f is integrable on $[a, b]$ for $\varepsilon > 0$ there is are step functions φ, ψ such that

$$\varphi \leq f \leq \psi$$

and

$$\int_a^b (\psi - \varphi)\, dx < \frac{\varepsilon}{2B}.$$

By replacing φ by $\max\{0, \varphi\}$ and ψ by $\min\{\psi, B\}$ we can assume $0 \leq \varphi$ and $\psi \leq B$. Then φ^2 and ψ^2 are step functions and

$$\varphi^2 \leq f^2 \leq \psi^2$$

and

$$0 \leq \psi^2 - \varphi^2 = (\psi + \varphi)(\psi - \varphi) \leq (\psi + \psi)(\psi - \varphi) \leq (B + B)(\psi - \varphi).$$

You should now be able to show

$$\int_a^b (\psi^2 - \varphi^2)\, dx < \varepsilon$$

so that Proposition 4 applies. □

Proposition 20. *If f and g are integrable on $[a, b]$ then so is the product fg.*

Problem 9. Prove this. *Hint:* Show

$$fg = \frac{(f + g)^2 - (f - g)^2}{4}$$

and use Lemma 19. □

2. The Fundamental Theorem of Calculus.

Proposition 21. *If $a < b < c$ and f is integrable on $[a, c]$ then the restrictions $f\big|_{[a,b]}$ and $f\big|_{[b,c]}$ are integrable on $[a, b]$ and $[b, c]$ respectively and*

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Proof. We have shown in class that for any bounded function on $[a, c]$ that

$$\overline{\int_a^c} f(x))\, dx = \overline{\int_a^b} f(x)\, dx + \overline{\int_b^c} f(x)\, dx,$$

$$\underline{\int_a^c} f(x))\, dx = \underline{\int_a^b} f(x)\, dx + \underline{\int_b^c} f(x)\, dx.$$

As f is integrable on $[a,c]$

$$\begin{aligned}
\int_a^c f(x)\,dx &= \overline{\int_a^c} f(x)\,dx \\
&= \underline{\int_a^c} f(x)\,dx \\
&= \underline{\int_a^b} f(x)\,dx + \underline{\int_b^c} f(x)\,dx \\
&\le \overline{\int_a^b} f(x)\,dx + \overline{\int_b^c} f(x)\,dx \\
&= \overline{\int_a^c} f(x)\,dx \\
&= \int_a^c f(x)\,dx.
\end{aligned}$$

Thus equality must hold at all the intermediate inequalities. Therefore

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx \quad \text{and} \quad \underline{\int_b^c} f(x)\,dx = \overline{\int_b^c} f(x)\,dx$$

which implies the restrictions $f\big|_{[a,b]}$ and $f\big|_{[b,c]}$ are integrable. The rest follows from

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx \quad \text{and} \quad \int_b^c f(x)\,dx = \overline{\int_b^c} f(x)\,dx$$

and that equality holds in the displayed inequality. □

Proposition 22. *Let f be integrable on $[a,b]$ and let $[\alpha,\beta]\subseteq[a,b]$. The f is integrable on $[\alpha,\beta]$.*

Problem 10. Prove this. *Hint:* $[\alpha,\beta]=[a,\beta]\cap[\alpha,b]$ and Proposition 21. □

It is useful to define $\int_a^b f(x)\,dx$ even in the cases where $a=b$ and $b<a$.

Definition 23. For any function f define

$$\int_a^a f(x)\,dx = 0.$$

If $b<a$ and f is integrable on $[b,a]$ define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

□

Proposition 24. *If f is integrable on the interval $[x_1,x_2]$ and $a,b,c\in[x_1,x_2]$ then, with the definitions above,*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof. This is just checking case by case (i.e. $a \le b \le c$, $a \le c \le b$ etc.) and is left to the reader. And please do not hand it in. $\square$

Proposition 25. *Let $f(x)$ be integrable on $[a,b]$ and let $F\colon [a,b] \to \mathbb{R}$ be defined by*

$$F(x) = \int_a^x f(t)\,dt$$

then F is Lipschitz. That is there is a constant M such that for all $x_1, x_2 \in [a,b]$,

$$|F(x_2) - F(x_1)| \le M|x_2 - x_1|$$

and therefore F is continuous on $[a,b]$.

Problem 11. Prove this. *Hint:* As f is integrable on $[a,b]$, it is bounded on $[a,b]$, say $|f(x)| \le M$ on $[a,b]$. Without loss of generality we can assume that $x_1 \le x_2$. Then

$$|F(x_2) - F(x_1)| = \left| \int_a^{x_2} f(t)\,dt - \int_a^{x_1} f(t)\,dt \right| = \left| \int_{x_1}^{x_2} f(t)\,dt \right| \le \int_{x_1}^{x_2} |f(t)|\,dt$$

and it should be easy from here. $\square$

Theorem 26 (Fundamental Theorem of Calculus Form 1)**.** *Let f be integrable on $[a,b]$. Define new function $F\colon [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\,dt.$$

If f is continuous at the point $x \in (a,b)$, then the derivative of F exists at x and

$$F'(x) = f(x).$$

Problem 12. Prove this. *Hint:* First note

$$1 = \frac{1}{h} \int_x^{x+h} 1\,dt.$$

Multiply by $f(x)$ to get

$$f(x) = \frac{1}{h} \int_x^{x+h} f(x)\,dt$$

Also note

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

Combining some of these formulas we get

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} - f(x) &= \frac{1}{h} \int_x^{x+h} f(t)\,dt - \frac{1}{h} \int_x^{x+h} f(x)\,dt \\ &= \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\,dt. \end{aligned}$$

Let $\varepsilon > 0$. As f is continuous at x there is a $\delta > 0$ such that

$$|t - x| < \delta \quad \implies \quad |f(t) - f(x)| < \varepsilon.$$

Put this all together to show

$$|h| < \delta \quad \implies \quad \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| < \varepsilon$$

and explain why this shows $F'(x) = f(x)$. □

Theorem 27 (Fundamental Theorem of Calculus Form 2)**.** *Let f be continuous on $[a, b]$ and let F be continuous on $[a, b]$ and differentiable (a, b) with $F' = f$ on (a, b). Then*

$$\int_a^b f(t)\,dt = F(b) - F(a) = F\Big|_a^b.$$

Problem 13*.* Prove this. *Hint:* Let

$$G(x) = \int_a^x f(t)\,dt - F(x)$$

and show $G'(x) = 0$ for $x \in (a, b)$. □

Corollary 28. *If f is continuous on $[a, b]$ and F is any anti-derivative of f on $[a, b]$ (that is $F'(x) = f(x)$ for $x \in [a, b]$), then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Problem 14*.* Prove this. □

Definition 29. Let f be integrable on $[a, b]$. Then the ***average value*** of f on $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\,dx.$$ □

Theorem 30 (The First Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then it achieves its average value. That is there is a $\xi \in (a, b)$ with*

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Problem 15*.* Prove this. *Hint:* As f is continuous on the closed bounded set $[a, b]$, it achieves its maximum and minimum on this interval. Let $m = \min\{f(x) : x \in [a, b]\}$ and $M = \max\{f(x) : x \in [a, b]\}$ and let $\alpha, \beta \in [a, b]$ such that $f(\alpha) = m$ and $f(\beta) = M$. Now

$$f(\alpha) = m = \frac{1}{b-a} \int_a^b m\,dx \leq \frac{1}{b-a} \int_a^b f(x)\,dx$$

and

$$f(\beta) = M = \frac{1}{b-a} \int_a^b M\,dx \geq \frac{1}{b-a} \int_a^b f(x)\,dx$$

and recall the intermediate value theorem. □

We now prove a somewhat stronger version of the second form of the Fundamental Theorem of Calculus.

Theorem 31. *Let F be continuous on $[a,b]$ assume that F is differentiable on (a,b) and let*

$$f(x)=F'(x)$$

on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

(This differs from Theorem 27 as we are only assuming that f is integrable rather than continuous.)

Proof. Let $\varepsilon>0$. As f is integrable there are step functions φ and ψ on $[a,b]$ with

$$(1)\quad \varphi\le f\le\psi \quad\text{and}\quad \int_a^b f\,dx-\varepsilon\le\int_a^b\varphi\,dx\le\int_a^b\psi\,dx\le\int_a^b f\,dx+\varepsilon.$$

We can assume there is a partition $\mathcal{P}=\{x_0,x_1,\dots,x_n\}$ such that if $I_j=[x_{j-1},x_j)$ then

$$\varphi=\sum_{j=1}^n m_j\chi_{I_j},\qquad \psi=\sum_{j=1}^n M_j\chi_{I_j}.$$

We write $F(b)-F(a)$ as a telescoping sum:

$$F(b)-F(a)=F(x_n)-F(x_0)=\sum_{j=1}^n\left(F(x_j)-F(x_{j-1})\right)$$

As F is differentiable on $[x_{j-1},x_j]$ we can apply the mean value theorem to get that there is a $\xi_j\in(x_{j-1},x_j)$ with

$$F(x_j)-F(x_{j-1}=F'(\xi_j)(x_j-x_{j-1})=f(\xi_j)(x_j-x_{j-1})=f(\xi_j)|I_j|.$$

Combining these equations gives

$$F(b)-F(a)=\sum_{j=1}^n\left(F(x_j)-F(x_{j-1})=\sum_{j=1}^n f(\xi_j)|I_j|.\right.$$

But $\varphi\le f\le\psi$ which implies $m_j\le f(\xi_j)\le M_j$ and thus

$$\int_a^b\varphi\,dx=\sum_{j=1}^n m_j|I_j|\le F(b)-F(a)=\sum_{j=1}^n f(\xi_j)|I_j|\le\sum_{j=1}^n M_j|I_j|=\int_a^b\psi\,dx.$$

Combining this with the inequalities (1) gives

$$\int_a^b f\,dx-\varepsilon\le F(b)-F(a)\le\int_a^b f\,dx+\varepsilon.$$

As $\varepsilon>0$ was arbitrary this gives $F(b)-F(a)=\int_a^b f\,dx$ as required. □

Problem 16. To see that Theorem 31 really is stronger than Theorem 27 we need to show that there is a function F on an interval $[a, b]$ such that $f = F'$ exists and is integrable on (a, b) but with f not continuous on (a, b). Let

$$F(x) = \begin{cases} x^2 \cos(1/x), & x \neq 0; \\ 0, & x = 0 \end{cases}$$

Show that F is differentiable at all points of $\mathbb{R}$, and $f = F'$ is bounded on $[-1, 1]$, but f is not continuous at $x = 0$. As f is continuous at all points other than 0 it is integrable on $[-1, 1]$. □

We can now give the familiar integration by parts formula.

Theorem 32 (Integration by Parts)**.** *Let u and v continuous on $[a, b]$, differentiable on (a, b), with u' and v' integrable on $[a, b]$. Then*

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{b} - \int_a^b u'(x)v(x)\,dx.$$

Problem 17. Prove this. *Hint:* This follows from the product rule and the Fundamental Theorem of Calculus in the form

$$\int_a^b (u(x)v(x))'\,dx = u(x)v(x)\Big|_{x=a}^{b}.$$

You do have to worry a bit about if the integrals involved exist. Theorem 20 should help here. □

We now use integration by parts to give another form of the remainder in Taylor's Theorem.

Lemma 33. *Let f be $k+1$ times differentable on an open interval (α, β) and assume that $f^{(k+1)}$ is integrable. Then for $a, x \in (\alpha, \beta)$ we have*

$$\int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,dt = \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt.$$

Problem 18. Prove this. *Hint:* Use integration by parts with $v'(t) = \frac{(x-t)^{k-1}}{(k-1)!}$ and $u = f^{(k)}(t)$. □

Theorem 34 (Taylor's Theorem with Integral form of the Remainder)**.** *Let f be $n+1$ times differentable on (α, β) and assume that $f^{(n+1)}$ is integrable. Then for $a, x \in (\alpha, \beta)$*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder term $R_n(x)$ is given by

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

Problem 19. Prove this. *Hint:* Note that Lemma 33 can be rewritten as

$$R_{k-1}(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

and by the Fundamental Theorem of Calculus and integration by parts

$$\begin{aligned} f(x) - f(a) &= \int_a^x f'(t)\,dt \\ &= -\int_a^x (-1)f'(t)\,dt \\ &= -\int_a^x \left(\frac{d}{dt}(x-t)\right) f'(t)\,dt \\ &= -\frac{d}{dt}(x-t)f'(t)\Big|_{t=a}^{x} + \int_a^x (x-t)f''(t)\,dt \\ &= f(a)(x-a) + R_1(x). \end{aligned}$$

Now use induction. □

Theorem 35 (Change of Variable Formula)**.** *Let the map $x = u(t)$ map the interval $[c,d]$ into the interval $[a,b]$ and assume that $u'(t)$ is integrable on $[c,d]$. Then for any continuous function f on $[a,b]$*

$$\int_{u(c)}^{u(d)} f(x)\,dx = \int_c^d f(u(t))u'(t)\,dt.$$

Problem 20. Prove this. *Hint:* Do this in steps

(a) Explain why both the integrals exist.

(b) Define F on $[a,b]$ by

$$F(x) = \int_a^x f(y)\,dy$$

and explain why

$$F'(x) = f(x) \qquad \text{and} \qquad \int_{u(c)}^{u(d)} f(x)\,dx = F(u(d)) - F(u(c)).$$

(c) On $[c,d]$ define

$$G(t) = F(u(t)).$$

By the chain rule

$$G'(t) = F'(u(t))u'(t) = f(u(t))u'(t)$$

and so by Theorem 31

$$\int_c^d f(u(t))u'(t)\,dt = \int_c^d G'(t)\,dt = G(d) - G(c).$$

(d) Put the pieces above together to finish the proof. □

3. Definition of the logarithm and exponential functions.

Define a function $L\colon (0,\infty) \to \mathbb{R}$ by

$$L(x) = \int_1^x \frac{dx}{x}.$$

We know this should be the natural logarithm, but we now verify directly from its definition that it has the correct properties.

Proposition 36. *The derivative of L is*

$$L'(x) = \frac{1}{x}$$

and thus L is strictly increasing. Therefore L is one-to-one (that is injective).

Proof. By the Fundamental Theorem of Calculus

$$L'(x) = \frac{1}{x} > 0$$

as $x > 0$ which implies L is strictly increasing. □

Proposition 37. *Let $a, b > 0$ then*

$$\int_a^b \frac{dx}{x} = L(b/a).$$

Problem 21. Prove this. *Hint:* In the integral $\int_a^b \frac{dx}{x}$ do the change of variable $x = at$ to get

$$\int_a^b \frac{dx}{x} = \int_1^{b/a} \frac{dt}{t}.$$ □

Proposition 38. *If $a, b > 0$ then*

$$L(ab) = L(a) + L(b).$$

Problem 22. Prove this. *Hint:*

$$L(ab) = \int_1^{ab} \frac{dx}{x} = \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x}$$

and use Proposition 37. □

The last Proposition and induction yield:

Corollary 39. *If $a > 0$ and n is a positive integer*

$$L(a^n) = nL(a).$$ □

Proposition 40. *The function $L\colon (0,\infty) \to \mathbb{R}$ is a bijection.*

Problem 23. Prove this. *Hint:* Recall the saying that L is a bijection is just saying that it is one-to-one and onto. We have already seen that L is injective. To see that it is surjective (that is onto) note that $L(2) > 0$ and $L(1/2) < 0$. Also for a positive integer n

$$L(2^n) = nL(2) \qquad \text{and} \qquad L(1/2^n) = nL(1/2).$$

If y_0 is any real number we can find (by Archimedes' principle) a positive integer n such that

$$nL(1/2) < y_0 < nL(2).$$

Also we know that L is continuous (why?). Now you should be able to show that there is a $x_0 \in (0, \infty)$ with $L(x_0) = y_0$. □

Because the function $L\colon (0, \infty) \to \mathbb{R}$ is bijective, it has an inverse $E\colon \mathbb{R} \to (0, \infty)$. As L is strictly increasing, continuous, and differentiable with $L'(x) \neq 0$ for all x theorems from earlier this term imply that E is strictly increasing, continuous, and differentiable.

Proposition 41. *The function E satisfies $E(0) = 1$ and*

$$E'(x) = E(x).$$

Problem 24. Prove this. *Hint:* $L(1) = 0$. And as L and E are inverses of each other $L(E(x)) = x$ for all $x \in \mathbb{R}$. Therefore $\frac{d}{dx} L(E(x)) = 1$. Use the chain rule and that we know the derivative of L. □

Proposition 42. *For all real numbers x*

$$E(-x) = \frac{1}{E(x)}.$$

Problem 25. Prove this. *Hint:* There are several ways to do this. One is to take the derivative of $E(x)E(-x)$ and show it is zero. Anther is to note that $L(a) + L(1/a) = L(1) = 0$ □

Proposition 43. *For all real numbers a, b*

$$E(a + b) = E(a)E(b).$$

Problem 26. Prove this. *Hint:* One way is to deduce this from the property $L(\alpha\beta) = L(\alpha) + L(\beta)$ of L. Anther is to show that the derivative of the function

$$f(x) = E(x + a)E(-x)$$

is zero and therefore f is constant. □

Proposition 44. *If n is any integer, positive or negative, and t is any real number*

$$E(nt) = E(t)^n$$

If m is a positive integer then

$$E\left(\frac{1}{m} t\right)^m = E(t)$$

and thus $E(\frac{1}{m} t)$ is the positive m-th root of $E(t)$.

Problem 27. Prove this. □

In light of Proposition 44 If r is a rational number, say $r = n/m$ with m, n integers and $m > 0$, then for a positive number a we can define

$$a^r = a^{n/m} = (a^n)^{1/m}$$

where $(a^n)^{1/m}$ is the positive m-th root of a^n. We would also like to define a^r when r is irrational. Note that when $r = m/n$ and $a = E(t)$, then Proposition 44 shows us that

$$a^r = E(t)^{n/m} = (E(t)^n)^{1/m} = E(nt)^{1/m} = E\left(\frac{1}{m}nt\right) = E(rt). \tag{2}$$

But $E(rt)$ makes sense for all real numbers r. We now formalize all this.

Definition 45. We now officially define ***logarithm*** of a positive number x to be

$$\ln(x) = L(x) = \int_1^x \frac{dt}{t},$$

the number e to be

$$e = E(1)$$

and for any real number x we define the power e^x by

$$e^x = E(x).$$

□

Definition 46. Let $a > 0$. Then for any real number r define

$$a^r = e^{r\ln(a)}.$$

(Note if $a = E(t) = e^t$ then $\ln(a) = t$ and this becomes $a^r = e^{r\ln(a)} = e^{rt} = E(rt)$ which agrees with our preliminary definition (2).) □

Proposition 47. *If $a > 0$ and $r = n/m$ is a rational number with $m > 0$, then*

$$a^r = (a^n)^{1/m}$$

so that our definition agrees with what it should be on the rational numbers. In particular $a^{1/2}$ is the square root of a, $a^{1/3}$ is the cube root of a etc.

Problem 28. Prove this. □

Proposition 48. *With these definition the following hold*

(a) *If $a > 0$ then for all $r, s \in \mathbb{R}$*

$$a^r a^s = a^{r+s}, \quad \frac{a^r}{a^s} = a^{r-s}.$$

and,

$$(a^r)^s = a^{rs}.$$

(b) *If $r \in \mathbb{R}$ and $a, b > 0$ then*

$$a^r b^r = (ab)^r.$$

(c) If $r, s \in \mathbb{R}$ and $a > 0$, then

$$(a^r)^s = a^{rs}.$$

Problem 29. Prove this. □

Proposition 49. *Let r be a real number and on define $f\colon (0,\infty) \to (0,\infty)$ by*

$$f(x) = x^r.$$

Then f is differentiable and

$$f'(x) = rx^{r-1}.$$

Problem 30. Prove this. *Hint:* We know that $E(x) = e^x$ is differentiable with derivative $E'(x) = E(x)$ and that $\ln(x)$ is differentiable with $\frac{d}{dx}\ln(x) = 1/x$. Thus $f(x) = e^{r\ln(x)} = E(r\ln(x))$ is a composition of differentiable functions. Use the chain rule to derive the formula for $f'(x)$. □

Proposition 50. *Let a be a positive real number and define $g\colon \mathbb{R} \to (0,\infty)$ by*

$$g(x) = a^x.$$

Then g is differentiable and

$$g'(x) = \ln(a)a^x.$$

Problem 31. Prove this. □

There is is anther way define e^x based on the following

Proposition 51. *For any $x \in \mathbb{R}$*

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Problem 32. Here is one method of proving this.

(a) Use Taylor's theorem with the Lagrange form of the remainder to show that for $|y| \le 1/2$ that

$$\ln(1+y) = y + R(y)$$

where

$$|R(y)| \le 2y^2.$$

(b) Let $x \in \mathbb{R}$ and note that if $|x/n| \le 1/2$, we have

$$\ln(1 + x/n) = \frac{x}{n} + R(x/n)$$

and

$$|R(x/n)| \le \frac{2x^2}{n^2}$$

(c) Use (b) to show

$$\lim_{n\to\infty} n\ln(1+x/n) = x.$$

(d) Now use that

$$\left(1+\frac{x}{n}\right)^n = e^{n\ln(1+x/n)}$$

to show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

holds. □

There are books that instead of defining the $\ln(x)$ as $\int_1^x dt/t$, and then defining e^x as the inverse of $\ln(x)$, first define e^x as

$$e^x = \left(1+\frac{x}{n}\right)^n,$$

show that this behaves like e^x should and then define $\ln(x)$ as the inverse of e^x and finally $a^r = e^{r\ln(a)}$. This takes more work than what we have done, but has the advantage that it is possible to define e^x, $\ln(x)$, and a^x before defining the derivative and integral.