Math 555 Homework.

Here are some problems for over Spring break.

Problem 1. Compute the following limits:

(a) $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n}=0$. *Hint:* For $x>0$ we have $\sqrt{x}<x$ and thus

$$\ln(n)=\int_1^n\frac{dx}{x}<\int_1^n\frac{dx}{\sqrt{x}}=2\sqrt{n}-2.$$

(b) For $\alpha>0$, $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n^\alpha}=0$. That is $\ln(n)$ grows more slowly than any power of n. *Hint:*

$$\frac{\ln n}{n^\alpha}=\frac{\ln(n^\alpha)}{\alpha n^\alpha}$$

and use the limit of part (c). □

Problem 2. (a) Let k be a positive integer. Show for all $x>0$

$$e^x>\frac{x^k}{k!}.$$

Hint: Taylor's Theorem.

(b) Let $r,\alpha>0$. Show

$$\lim_{n\to\infty}\frac{n^\alpha}{e^{rn}}=0.$$

That is e^{rn} grows faster than any power of n. *Hint:* From Part (a) we have that for any positive integer k,

$$e^{rn}\geq\frac{r^kn^k}{k!}$$

and so

$$0<\frac{n^\alpha}{e^{rn}}<\frac{k!n^\alpha}{r^kn^k}.$$

We can use any k we want, so choose $k>\alpha$. □

Problem 3. In this problem we show that the integration by parts trick we used to prove Taylor's Theorem can be used to prove other interesting results. Let k be an integer and let

$$P_0(x)=1$$
$$P_1(x)=x-(k+1/2)$$
$$P_2(x)=\frac{(x-k)(x-k-1)}{2}.$$

Let $f(x)$ be a function on $[k,k+1]$ that is twice continuously differentiable.

(a) Show

$$P_1'(x)=1$$
$$P_2'(x)=P_1(x).$$

(b) Integrate

$$\int_k^{k+1} f(x)\,dx = \int_k^{k+1} P_1'(x) f(x)\,dx$$

by parts twice to get

$$\int_k^{k+1} f(x)\,dx = P_1(x)f(x)\Big|_k^{k+1} - P_2(x)f'(x)\Big|_k^{k+1} + \int_k^{k+1} P_2(x)f''(x)\,dx.$$

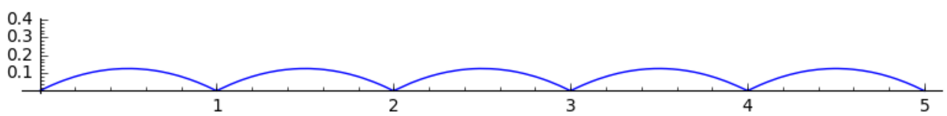
(c) Show this simplifies down to

$$\int_k^{k+1} f(x)\,dx = \frac{f(x)+f(k+1)}{2} + \int_k^{k+1} P_2(x)f''(x)\,dx$$

(d) Now define a function $B\colon \mathbb{R} \to \mathbb{R}$ by

$$B(x) = \frac{(x-k)(k+1-x)}{2} = -P_2(x) \quad \text{when} \quad k \le x \le k+1.$$

This function is periodic with period 1, that is $B_2(x+1) = B(x)$. The graph looks like



Show that with this notation we have

$$\frac{f(k)+f(k+1)}{2} = \int_k^{k+1} f(x)\,dx + \int_k^{k+1} B(x)f''(x)\,dx.$$

Also show

$$0 \le B(x) \le \frac{1}{8}.$$

(The advantage of working with $B(x)$ is that it is positive, and so it is easier to keep track of signs.)

(e) Now sum the equality for $(f(k)+f(k+1))/2$ from $k=1$ to $n-1$ and rearrange a bit to get

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{f(1)+f(n)}{2} + \int_1^n B(x)f''(x)\,dx. \tag{1}$$

This gives a precise relation between sums and integrals of the same function and is a special case of the ***Euler-Maclaurin Summation Formula***.

Problem 4. To give a concrete application of the formula (1) we let $f(x) = \ln(x)$ and derive a very useful approximation to $n!$ (***Stirling's Formula***).

(a) With this choice of $f(x)$ show that (1) becomes

$$\begin{aligned}\ln(n!) &= \sum_{k=1}^{n} \ln(k) \\ &= \int_1^n \ln(x)\,dx + \frac{\ln(1)+\ln(n)}{2} + \int_1^n B(x)\ln''(x)dx \\ &= n\ln(n) - n + \frac{\ln(n)}{2} - \int_1^n \frac{B(x)}{x^2}\,dx\end{aligned}$$

(b) Note that

$$0 < \int_1^n \frac{B(x)}{x^2}\,dx \;< \int_1^n \frac{1}{8x^2}\,dx = \frac{1}{8}\left(1 - \frac{1}{n}\right) < \frac{1}{8}$$

and use this to show that

$$C := \lim_{n\to\infty} \int_1^n \frac{B(x)}{x^2}\,dx$$

exists and

$$0 < C \leq \frac{1}{8}.$$

(c) Rewrite the formula for $\ln(n!)$ as

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + C - R_n$$

where

$$R_n = C - \int_0^n \frac{B(x)}{x^2}\,dx.$$

satisfies

$$0 < R_n \leq \frac{1}{8n}.$$

(d) Use this to conclude

$$n! = e^C n^{n+1/2} e^{-n} e^{-R_n}.$$

Setting $K = e^C$ we then have

$$Kn^{n+1/2}e^{-n}e^{-1/(8n)} < n! < Kn^{n+1/2}e^{-n}.$$

Or in slightly different form

$$e^{-1/(8n)} < \frac{n!}{Kn^{n+1/2}e^{-n}} < 1.$$

Thus $n!$ has the same growth rate as $Kn^{n+1/2}e^{-n} = K\sqrt{n}\left(\frac{n}{e}\right)^n$. Of course we would like to know the constant K. It turns out

$$K = \sqrt{2\pi},$$

a fact that hopefully we will be able to show by the end of the term. Using this we can rewrite out inequalities as

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-1/8n} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$\square$