Mathematics 554 Homework.

(1) For some review of limits in *Notes on Analysis II* Section 1 do Problems 1.1, 1.2.

(2) In Section 2 do problem (which is to show that $f(x) = \sqrt{x}$ is differentiable at all points $a > 0$).

(3) If we let $y = x^{\frac{1}{3}}$ and $b = a^{\frac{1}{3}}$ in the identity $y^3 - b^3 = (y-b)(y^2 + yb + b^2)$ we get

$$x - a = (x^{\frac{1}{3}} - a^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}).$$

so that

$$\frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x - a} = \frac{1}{x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}}.$$

Last term we showed that the functions $x \mapsto x^{\frac{1}{3}}$ and $x \mapsto x^{\frac{2}{3}}$ are continuous. Use these facts to show $f(x) = x^{\frac{1}{3}}$ is differentiable at all $x = a \neq 0$ and that $f'(a) = \frac{1}{3}a^{\frac{-2}{3}}$.

(4) Do a variant on the previous problem to show $f(x) = x^{\frac{1}{4}}$ is differentiable at all points $a > 0$ and that $f'(a) = \frac{1}{4}a^{\frac{-3}{4}}$. (Or if you wish you generalize this to showing that $f(x) = x^{\frac{1}{n}}$ is differentiable at all points $a > 0$ for an positive integer n.)

(5) Problem 2.6 (that is prove the product rule).

(6) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}; \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers.

(a) Show that for any $a \neq 0$ that f is not continuous at a.

(b) Show f is differentiable at $x = 0$.