Mathematics 554 Homework.

Let us take a break from the Riemann integral to cover a topic more related to what we did last term.

Definition 1. Let $f\colon X \to Y$ be a map between metric spaces. Then f is ***uniformly continuous*** if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $p_1, p_2 \in X$,

$$d_X(p_1, p_2) < \delta \quad \text{implies} \quad d_Y(f(p_1), f(p_2)) < \varepsilon. \qquad \square$$

Recall the definition of f being continuous:

Definition 2. Let $f\colon X \to Y$ be a map between metric spaces. Then f is ***continuous*** if and only if for all $p_0 \in X$ and $\varepsilon > 0$ there is is a $\delta > 0$ such that for all $p \in X$,

$$d_X(p_0, p) < \delta \quad \text{implies} \quad d_Y(f(p_0), f(p)) < \varepsilon. \qquad \square$$

Thus for f to be continuous the δ depends on both p_0 and ε. Then f is uniformly continuous we can choose δ so which only depend on ε.

For an example of a function that is continuous, but not uniformly continuous let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) = x_2$. Towards a contradiction assume f is uniformly continuous. Let $\varepsilon = 1$. Then there would be a $\delta > 0$ so that for all $x_1, x_2 \in \mathbb{R}$

$$|x_2 - x_1| < \delta \quad \text{implies} |f(x_2) - f(x_1)| = |x_2^2 - x_1^2| < \varepsilon = 1.$$

Let $x_1 = x$ and $x_2 = x + \delta$, then $|x_2 - x_1| = \delta/2 < \delta$ and therefore for all x

$$|(x + \delta/2)^2 - x^2| = \delta|x + \delta/4| \leq 1$$

and therefore

$$|x_\delta/4| \leq \frac{1}{\delta}.$$

But this does not hold for all $x \in \mathbb{R}$ and so f is not uniformly continuous.

Problem 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then there are constants M and B so that

$$|f(x)| \leq M|x| + B$$

for all $x \in \mathbb{R}$. *Hint:* You should consider this a challenge problem as it is harder than most. $\square$

Recall that a metric space X is compact if and only if every open cover of X has a finite sub-cover.

Here is the result we will be needing to show that the integral of a continuous function exists.

Theorem 3. *Let $f\colon X \to Y$ be a continuous function between metric spaces with X compact. Then f is uniformly continuous.*

Problem 2. Prove this. *Hint:* Here is an outline of one way to do this. Let $\varepsilon > 0$. Then as f is continuous for each $p \in X$ there is a $\delta_p > 0$ so that

$$d_X(p,q) < \delta_p \quad \text{implies} \quad d_Y(f(p), f(q)) < \frac{\varepsilon}{2}.$$

Then the collection of open balls

$$\mathcal{U} : B(p, \delta_p/2 : p \in X\}$$

is an open cover of X. (We did enough of this type of construction last term that you do not need to prove this is an open cover.) Thus $\mathcal{U}_0$ has a finite subset set that covers X. Let

$$\mathcal{U}_0 = \{B(p_1, \delta_1/2), B(p_2, \delta_2/2), \ldots, B(p_n, \delta_n/2)\}$$

where we have simplified notation by setting $\delta_j = \delta_{p_j}$. Let

$$\delta = \min\{\delta_1/2, \delta_2/2, \ldots, \delta_n/2\}.$$

Let $x_1, x_2 \in X$ with

$$d_X(x_1, x_2) < \delta.$$

(a) As $\mathcal{U}_0$ is an open cover of X there is j so that $x_1 \in B(p_j, \delta_j/2)$. Use $d_X(x_1, x_2) < \delta \le \delta_j/2$ to show $d_X(x_2, p_j) < \delta_j$.
(b) Use $d_X(x_1, x_2) \le d_X(x_1, p_j) + d_X(p_j, x_2)$ to show $d_Y(f(x_1), f(x_2)) < \varepsilon$.
(c) Explain why we are done. □

Problem 3. Do Problems 1.3 and 1.4 on page 3 in the notes. □