Mathematics 554 Homework.

These problems will not be collected, but should prepare you for the test on Wednesday and hopefully at least a couple of them are interesting.

Problem 1. Let $f\colon [a,b] \to \mathbb{R}$ be a function such that for some constants $M > 0$ and $\alpha > 0$

$$|f(x_2) - f(x_1)| \le M|x_2 - x_1|^\alpha$$

for all $x_1, x_2 \in [a,b]$. (We say that f satisfies a ***Hölder condition*** of order α.) Show that if f satisfies a Hölder condition with $\alpha > 1$ that $f'(x) = 0$ for all $x \in (a,b)$ and thus f is constant. (If you do not want to work with a general α it is enough do the problem in the case where $\alpha = 3/2$.) □

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the differential equation

$$f'(x) = \frac{f(x)^2}{1 + f(x)^4}.$$

Show f is a monotone increasing function. □

Problem 3. Let y satisfy the differential equation

$$y'(x) = \frac{x^2 - 1}{1 + x^2 y(x)^2}.$$

Show that y has a local maximum at $x = -1$ and a local minimum at $x = 1$. □

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with $|1 - f'(x)| < \rho$ where ρ is a constant with $\rho < 1$. Let $g(x) = x - f(x)$.

(a) Show $f(x) = 0$ if and only if $g(x) = x$.

(b) Let r be a point where $f(r) = 0$, and thus $g(x_0) = x_0$. Use the Mean Value Theorem to show that for any x we have

$$|g(x) - r| = |g(x) - g(r)| < \rho|r|.$$

Hint: $g' = x - f'$.

(c) Let $x_0 \in \mathbb{R}$ and define a sequence by $x_1 = g(x_0)$, $x_2 = g(x_1)$ and in general $x_{n+1} = g(x_n)$. Show $\lim_{n\to\infty} x_n = r$.

Problem 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be four times differentiable with $f^{(4)}(x) > 0$ for all $x \in \mathbb{R}$. Assume that there is a point a with $f'(a) = f''(a) = f'''(a) = 0$. Show that f has a global minimum at $x = a$. □

Problem 6. Let f and g be n times differentiable on (a,b) and let $p = fg$ be the product of f and g. Note using the product rule it is not hard to see

$$\begin{aligned} p' &= f'g + fg' \\ p'' &= f''g + 2f'g' + fg'' \\ p''' &= f'''g + 3f''g' + 3f'g'' + g'''. \end{aligned}$$

Guess a formula for the n-th derivative $p^{(n)}$ and prove it is correct. □

Problem 7. Use that the derivative of sec is

$$\sec'(x) = \sec(x)\tan(x)$$

and that $1 + \tan^2(x) = \sec^2(x)$ to derive a formula for the derivative of $\operatorname{arcsec}(x)$.

Problem 8. Let $f, g\colon [a, b] \to \mathbb{R}$ be functions differentiable on (a, b), continuous on $[a, b]$ and with $f(a) = g(a)$ and $f(b) = g(b)$. Show there is a point $\xi \in (a, b)$ with $f'(\xi) = g'(\xi)$. □

Problem 9. Let f be twice differentiable on an interval containing a and assume f'' is continuous. Compute

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$