Math 555 Homework.

Here are some problems for over Spring break.

Problem 1. For the following series say if they converge or diverge and why.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+1}{3^k+5}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(2n)!}{(n!)^2}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{n!}{n^n}$.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$.

Problem 2. Since it came up in class let us prove the following version of l'Hôpital rule. Let $f, g\colon [0,\infty) \to \mathbb{R}$ be differentiable functions with

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to\infty} g(x) = \infty$$

Assume

$$\lim_{x\to\infty} \frac{f'(a)}{g'(x)} = L.$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(a) As in the version of l'Hôpital's we have proven, the main tool is the Cauchy mean value theorem. As $\lim_{x\to\infty} f'(x)/g'x(x) - L$ there is a $a > 0$ such that

$$x \ge a \quad \text{implies} \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$$

Use the Cauchy mean value theorem to show

$$x \ge a \quad \text{implies} \quad \left| \frac{f(x)-f(a)}{g(x)-g(a)} - L \right| < \varepsilon.$$

(b) Use that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ to show for any $a > 0$

$$\lim_{x\to\infty} \frac{f(x)}{f(x)-f(a)} = 1.$$

Likewise

$$\lim_{x\to\infty} \frac{g(x)-g(a)}{g(x)} = 1$$

(Just prove one of these, as the proofs are pretty much the same.)

(c) Noting

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \frac{f(x)}{f(x) - f(a)} \frac{g(x) - g(a)}{g(x)}$$

you should be able to finish the proof.

Problem 3. In this problem we show that the integration by parts trick we used to prove Taylor's theorem with the integral form of the remainder term can be used to prove other interesting results. Let k be an integer and let

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x - (k + 1/2) \\ P_2(x) &= \frac{(x-k)(x-k-1)}{2}. \end{aligned}$$

Let $f(x)$ be a function on $[k, k+1]$ that is twice continuously differentiable.

(a) Show

$$\begin{aligned} P_1'(x) &= 1 \\ P_2'(x) &= P_1(x). \end{aligned}$$

(b) Integrate

$$\int_k^{k+1} f(x)\,dx = \int_k^{k+1} P_1'(x) f(x)\,dx$$

by parts twice to get

$$\int_k^{k+1} f(x)\,dx = P_1(x)f(x)\Big|_k^{k+1} - P_2(x)f'(x)\Big|_k^{k+1} + \int_k^{k+1} P_2(x) f''(x)\,dx.$$

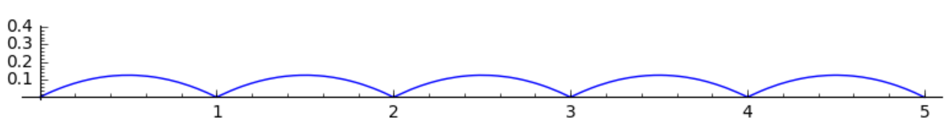
(c) Show this simplifies down to

$$\int_k^{k+1} f(x)\,dx = \frac{f(x) + f(k+1)}{2} + \int_k^{k+1} P_2(x) f''(x)\,dx$$

(d) Now define a function $B\colon \mathbb{R} \to \mathbb{R}$ by

$$B(x) = \frac{(x-k)(k+1-x)}{2} = -P_2(x) \quad \text{when} \quad k \le x \le k+1.$$

This function is periodic with period 1, that is $B_2(x+1) = B(x)$. The graph looks like



Show that with this notation we have

$$\frac{f(k) + f(k+1)}{2} = \int_k^{k+1} f(x)\,dx + \int_k^{k+1} B(x) f''(x)\,dx.$$

Also show

$$0 \le B(x) \le \frac{1}{8}.$$

(The advantage of working with $B(x)$ is that it is positive, and so it is easier to keep track of signs.)

(e) Now sum the equality for $(f(k)+f(k+1))/2$ from $k=1$ to $n-1$ and rearrange a bit to get

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{f(1)+f(n)}{2} + \int_1^n B(x)f''(x)\,dx. \tag{1}$$

This gives a precise relation between sums and integrals of the same function and is a special case of the ***Euler-Maclaurin Summation Formula***.

Problem 4. To give a concrete application of the formula (1) we let $f(x) = \ln(x)$ and derive a very useful approximation to $n!$ (***Stirling's Formula***).

(a) With this choice of $f(x)$ show that (1) becomes

$$\begin{aligned}
\ln(n!) &= \sum_{k=1}^{n} \ln(k) \\
&= \int_1^n \ln(x)\,dx + \frac{\ln(1)+\ln(n)}{2} + \int_1^n B(x)\ln''(x)dx \\
&= (x\ln(x) - x)\Big|_1^n + \frac{\ln(n)}{2} - \int_1^n \frac{B(x)}{x^2}\,dx \\
&= n\ln(n) - n + 1) + \frac{\ln(n)}{2} - \int_1^n \frac{B(x)}{x^2}\,dx
\end{aligned}$$

(b) Note

$$0 < \int_1^n \frac{B(x)}{x^2}\,dx \; < \int_1^n \frac{1}{8x^2}\,dx = \frac{1}{8}\left(1 - \frac{1}{n}\right) < \frac{1}{8}$$

and use this to show

$$\int_0^\infty \frac{B(x)}{x^2}\,dx = \lim_{n\to\infty} \int_1^n \frac{B(x)}{x^2}\,dx$$

exists and

$$0 < \int_1^\infty \frac{B(x)}{x^2}\,dx \le \frac{1}{8}.$$

(c) Rewrite the formula for $\ln(n!)$ as

$$\begin{aligned}
\ln(n!) &= \left(n + \frac{1}{2}\right)\ln(n) - n + 1 - \int_1^\infty \frac{B(x)}{x^2}\,dx + \int_n^\infty \frac{B(x)}{x^2}\,dx \\
&\left(n + \frac{1}{2}\right)\ln(n) - n + C + R_n
\end{aligned}$$

where

$$C = 1 - \int_1^\infty \frac{B(x)}{x^2}, \quad \text{and} \quad R_n = \int_0^n \frac{B(x)}{x^2}\,dx$$

satisfies

$$0 < R_n \leq \frac{1}{8n}.$$

(d) Use this to conclude

$$n! = e^C n^{n+1/2} e^{-n} e^{R_n}.$$

Setting $K = e^C$ we then have

$$Kn^{n+1/2}e^{-n} < n! < Kn^{n+1/2}e^{-n}e^{\frac{1}{8n}}.$$

Or in slightly different form

$$1 < \frac{n!}{Kn^{n+1/2}e^{-n}} < e^{\frac{1}{8n}}.$$

Thus $n!$ has the same growth rate as $Kn^{n+1/2}e^{-n} = K\sqrt{n}\left(\frac{n}{e}\right)^n$. Of course we would like to know the constant K. It turns out

$$K = \sqrt{2\pi},$$

a fact you will show in the next problem. Using this we can rewrite out inequalities as

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{8n}}.$$

This is often written as

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$$

where the notation $f(n) \sim g(n)$ is shorthand for

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1. \qquad \square$$

Example 1. Here is an example of Stirling's formula in practice. The probability of flipping a fair coin $2n$ times and have the same number of heads and tails is

$$p_n = \binom{2n}{n}\frac{1}{2^{2n}} = \frac{(2n)!}{(n!)^2}\frac{1}{2^{2n}}.$$

While exact, this does not give a feel for the size of the number. But we can approximate.

$$\begin{aligned} p_n &= \frac{(2n)!}{(n!)^2}\frac{1}{2^{2n}} \\ &\sim \frac{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}}e^{-2n}}{(\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n})^2 2^{2n}} \\ &= \frac{1}{\sqrt{\pi n}}. \end{aligned}$$

This approximation is good. For $n = 50$, that is 100 coin tosses, the true value is $0.07958\ldots$ and the approximation just given is $0.07978\ldots$. $\square$

On the previous homework you have shown

$$I_n = \int_{-1}^{1} (1-x^2)^n \, dx$$

has the value

$$I_n = \frac{(n!)^2 2^{2n+1}}{(2n+1)!}$$

Let us approximate this by using a special case of what is called the "Method of Laplace". Do the change of variable $x = u/\sqrt{n}$ to get

$$I_n = \frac{1}{\sqrt{n}} \int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{u^2}{n}\right)^n du = \frac{C_n}{\sqrt{n}}$$

where

$$C_n = \int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{u^2}{n}\right)^n du$$

We have shown that

$$\lim_{n\to\infty} \left(1 - \frac{u^2}{n}\right)^n = e^{-u^2}$$

so it is very plausible, and not too hard prove, that

$$\lim_{n\to\infty} C_n = \int_{-\infty}^{\infty} e^{-u^2} \, du = \sqrt{\pi}.$$

Thus we have the asymptomatic formula

$$I_n \sim \frac{\sqrt{\pi}}{\sqrt{n}} \tag{2}$$

Problem 5. Use the asymptomatic formulas

$$I_n \sim \frac{\sqrt{\pi}}{\sqrt{n}} \qquad \text{and} \qquad n! \sim K n^{n+\frac{1}{2}} e^{-n}$$

to show

$$I_n \sim \frac{K}{\sqrt{2n}}$$

and thus conclude $K = \sqrt{2\pi}$. □

Problem 6. In *Notes on Analysis* do problems 1.4, 1.5, 1.6, 1.7, 1.8.