Mathematics 555 Homework.

Review for Test 2

Problem 1. For the basics about integrals you should know the definition of a step function and the definitions of the upper and lower integrals

$$\overline{\int_a^b} f(x)\,dx, \qquad \underline{\int_a^b} f(x)\,dx$$

and that f is integrable if and only if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Problem 2. You know know the basic linearity properties of the integral, that is if f and g are Riemannian integrable and α and β are real numbers then $\alpha f + \beta g$ is integrable and

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

While the proof of this is not hard, it is too long for me to give as part of an in class test.

Problem 3. Be able to prove that if $f\colon [a,b] \to \mathbb{R}$ is an increasing function, then $\int_a^b f(x)\,dx$ exists.

Problem 4. Other basic properties of the integral to know are

(a) If $f(x) \le g(x)$ for $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

and if $a \le b$ then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Also for any a, b, c

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

provided the integrals exist on the indicated intervals.

Problem 5. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and set

$$F(x) = \int_a^x f(t)\,dt.$$

Be able to use the basic properties of the integral to show F is differentiable at all points $x_0 \in (a,b)$ and

$$F'(x_0) = f(x_0).$$

(This is our first form of the Fundamental Theorem of Calculus.)

Problem 6. Be able to use the first form of the Fundamental Theorem of Calculus to prove that if f is continuous on $[a,b]$ and F is an anti-derivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is our second form of the Fundamental Theorem of Calculus.

Problem 7. You may have to compute an integral so know the basic tricks of the trade. That is u-substitution, integration by parts, etc.

Problem 8. Know our official definition of the natural logarithm:

$$\ln(x) = \int_1^x \frac{dt}{t}$$

which holds for $x > 0$. Be able to use this definition to prove

$$\ln(ab) = \ln(a) + \ln(b)$$

and

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Problem 9. Know our official definition of e^x as the inverse of ln. Be able to prove

$$\frac{d}{dx}e^x = e^x.$$

Be able to prove if $f(x)$ is a function that satisfies

$$f'(x) = cf(x),$$

then

$$f(x) = f(0)e^{cx}.$$

Problem 10. Some definitions and theorems you should know about series.

(a) For a series $\sum_{k=1}^{\infty} a_k$ know the definition of the n-th partial sum $A_n := \sum_{k=1}^{n} a_n$ and the definition of what it means for a series to converge, i.e. $\sum_{k=1}^{\infty} a_k = A$ if and only if $A = \lim_{n\to\infty} A_n$. (That is the sum of an infinite series is the limit of its partial sums.)

(b) Know that basic results about linearity of finite sums. For example be able to prove that if both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge and α and β are real numbers then $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ converges and

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \alpha\sum_{k=1}^{\infty} a_k + \beta\sum_{k=1}^{\infty} b_k.$$

(c) You should also be able to prove: if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Thus if $\lim_{k\to\infty} a_k \neq 0$, the series diverges.

Problem 11. You should know Taylor's Theorem with with Lagrange's form of the remainder.

Problem 12. You should know the basic convergence tests for series of positive terms as in Sections 6.1 to 6.3. Among these are the basic comparison and limit comparison tests. The integral test and how to use it to show $\sum_{k=1}^{\infty} 1/k^p$ converges if and only if $p > 1$. This also includes the root and ratio tests. I would consider having you state and prove the root test as fair game.

Problem 13. You should know the definitions absolutely and conditionally convergent series and an example of a conditionally convergent series. You should also know the alternating series test.

Problem 14. The most recent topic we have covered is power series. You should know the definition of the radius of convergence and how to find it. Our most recent theorem is that if

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$$

has radius of convergence R then the f is differentiable in the interval the open interval $(a-R, a+R)$ and its derivative is given by

$$f'(x) = \sum_{k=0}^{\infty} k c_k (x-a)^{k-1}.$$