NOTES ON ANALYSIS.

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Contents

1. Preliminaries.

In this section we review some algebra that will be useful later in the course.

1.1. Summation notation.

1.1. Summation notation. Summation notation will be used a great deal in this class. We recall the basics about it. The notation is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Thus

$$\sum_{k=0}^{5} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

There is nothing special about using k for the index:

$$\sum_{k=1}^{100} a_k = \sum_{j=1}^{100} a_j = \sum_{\alpha=1}^{100} a_\alpha = \sum_{\smiley=1}^{100} a_{\smiley} = \sum_{\frownie=1}^{100} a_{\frownie}.$$

A basic property of sums is

$$c_1 \sum_{k=m}^{n} a_k + c_2 \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (c_1 a_k + c_2 b_k).$$

We will also want to do changes of index in sum. For example

$$\begin{aligned}\sum_{k=m}^{n} a_k x^{k+3} &= a_m x^{m+3} + a_{m+1} x^{m+4} + \cdots + a_{n-1} x^{n+2} + a_n x^{n+3} \\ &= \sum_{k=m+3}^{n+3} a_{k-3} x^k.\end{aligned}$$

1.2. Geometric series. A (finite) ***geometric series*** is a finite sum of the form

$$S = a + ar + ar^2 + \cdots + ar^n.$$

In summation notation this is

$$S = \sum_{k=0}^{n} ar^k.$$

Such sums occur naturally in many contexts and fortunately it is easy give a formula for their sum. We first look at the case of $n = 2$. Then

$$S = a + ar + ar^2.$$

Multiply this by r to get

$$rS = ar + ar^2 + ar^3.$$

Note that the sums for S and rS have the terms ar and ar^2 in common, which suggests subtracting to cancel these terms out:

$$\begin{aligned} S &= a + ar + ar^2 \\ -rS &= \quad - ar - ar^2 - ar^3 \\ S - rS &= a - ar^3. \end{aligned}$$

Therefore

$$(1-r)S = a - ar^3$$

which, when $r \neq 1$, we can solve for S to get

$$S = \frac{a - ar^3}{1-r}$$

For $n = 5$ the calculation looks like

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 \\ -rS &= \quad - ar - ar^2 - ar^3 - ar^4 - ar^5 - ar^6 \\ S - rS &= a - ar^6 \end{aligned}$$

and therefore

$$(1-r)S = a - ar^6.$$

So when $r \neq 1$ we have

$$S = \frac{a - ar^6}{1-r}.$$

At this point you have likely guessed the general pattern:

Theorem 1.1. *Let a and r be real numbers with $r \neq 1$ and $n \geq 2$ and integer. Then the sum of the geometric series*

$$S = a + ar + ar^2 + \cdots + ar^n$$

is

$$S = \frac{a - ar^{n+1}}{1-r}.$$

Problem 1.1. Prove this. □

Problem 1.2. What happens in the theorem when $r = 1$? □

The way I find easiest to remember and apply this is to note that if the series $a + ar + ar^2 + \cdots + ar^n$ is continued that the next term would be ar^{n+1}. Therefore if we call the number r the ***ratio*** then

$$a + ar + ar^2 + \cdots + ar^n = \frac{1 - \text{next term}}{1 - \text{ratio}}.$$

Here are some examples

$$x^2 + x^4 + x^6 + \cdots + x^{20} = \frac{\text{first} - \text{next term}}{1 - \text{ratio}} = \frac{x^2 - x^{22}}{1 - x^2}$$

holds when $x \neq \pm 1$.

Let

$$S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}.$$

Then

$$\begin{aligned} S &= \frac{1 - \text{next term}}{1 - \text{ratio}} \\ &= \frac{1-(-1/128)}{1-(-1/2)} = \frac{128+1}{128+64} = \frac{129}{192}. \end{aligned}$$

Let

$$\alpha = \overbrace{.333\cdots 3}^{n \text{ digits}}.$$

Then

$$\begin{aligned} \alpha &= 3(.1) + 3(.1)^2 + 3(.1)^3 + \cdots + 3(.1)^n \\ &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{3(.1) - 3(.1)^{n+1}}{1 - .1} \\ &= \frac{.3 - .3(.1)^n}{.3(3)} \\ &= \frac{1}{3} - \frac{1}{3(10)^n} \end{aligned}$$

There is anther natural way to find α:

$$\begin{aligned} 9\alpha = 10\alpha - \alpha &= (3.33\cdots 3) - (.333\cdots 3) \\ &= 3 - \underbrace{.000\cdots 3}_{\text{n decmal places}} \end{aligned}$$

Therefore

$$\alpha = \frac{3 - .000\cdots 3}{9} = \frac{1}{3} - \frac{.000\cdots 1}{3} = \frac{1}{3} - \frac{1}{3(10)^n}$$

For the classical problem[1] of putting one grain rice on the first square of a chess broad, two on the second square, four on the third square, eight on the fourth square: that is doubling the number on each square up until the 64th square, then the total number of grains is

$$1 + 2 + 4 + \cdots + 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615.$$

Remark 1.2. The internet tells me that "A single long grain of rice weighs an average of 0.001 ounces (29 mg)." Thus the total weight of the rice on the chess board is $(2^{64}-1)/(1{,}000)$ onces. The number of onces $(2^{64}-1)/(1{,}000)$ in a ton is $2{,}000 \times 16 = 32{,}000$. Therefore the weight in tons of the rice

$$W = (2^{64} - 1)/(1{,}000 \times 32{,}000) = 5.76460752303423 \times 10^{11}.$$

The internet also says that the current rate of world rice production is about $P = 7.385477 \times 10^8$ tones/year. At this rate is would take about

$$\frac{W}{P} \approx 780.533$$

years to cover the chess board. □

Problem 1.3. (a) Find the sum of $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n}$

(b) Find the sum of $P_0(1 + r) + P_0(1 + r)^2 + \cdots + P_0(1 + r)^n$. (If at the beginning of each year you put P_0 in a bank account that pays interest at a rate of $100r\%$ per year, then this sum is the total after n years. As a check on your answer when $P_0 = 1{,}000$ and $r = .05$, (that is a 5% simple interest) then after 20 years the total is, to the nearest penny, 34,719.25.) □

1.3. **Some useful factoring formulas.** You recall that

$$x^2 - y^2 = (x - y)(x + y)$$

and may recall that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

[1] From the Wikipedia article on putting on grains of rice (or wheat) *Wheat and chessboard problem* `https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem` The problem appears in different stories about the invention of chess. One of them includes the geometric progression problem. The story is first known to have been recorded in 1256 by Ibn Khallikan.[1] Another version has the inventor of chess (in some tellings Sessa, an ancient Indian Minister) request his ruler give him wheat according to the wheat and chessboard problem. The ruler laughs it off as a meager prize for a brilliant invention, only to have court treasurers report the unexpectedly huge number of wheat grains would outstrip the ruler's resources. Versions differ as to whether the inventor becomes a high-ranking advisor or is executed.

These generalize. To see how let us look at the right hand side of the last equation. If we multiple this out we get

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2)-y(x^2+xy+y^2)\\ &= x^3+x^2y+xy^2-x^2y-xy^2-y^3 \qquad \text{(most terms cancel)}\\ &= x^3-y^3.\end{aligned}$$

Let us look at a similar product:

$$\begin{aligned}(x-y)(x^3+x^2y+xy^2+y^3) &= x(x^3+x^2y+xy^2+y^3)-y(x^3+x^2y+xy^2+y^3)\\ &= x^4+x^3y+x^2y^2+xy^3\\ &\qquad -x^3y-x^2y^2-xy^3-y^4\\ &= x^4-y^4.\end{aligned}$$

And just to be sure we see the pattern let us look at the next case

$$\begin{aligned}(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) &= x(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &\qquad -y(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &= x^5+x^4y+x^3y^2+x^2y^3+xy^4\\ &\qquad -x^4y-x^3y^2-x^2y^3-xy^4-y^5\\ &= x^5-y^5.\end{aligned}$$

The pattern is now clear:

Theorem 1.3. *Let n be any positive integer and let x and y be any two real numbers. Then*

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1}).$$

In summation notation this is

$$x^n-y^n=(x-y)\left(\sum_{k=0}^{n-1}x^{n-1-k}y^k\right)=(x-y)\left(\sum_{\substack{j+k=n-1\\0\le j,k\le n-1}}x^jy^k\right)$$

Problem 1.4. Prove this by multiplying out $(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$ and seeing that all but two terms cancel. □

Problem 1.5. The proof of Theorem 1.3 may remind you of the proof of Theorem 1.1 because both rely on a lot of cancellation. This is because there is a geometric series hidden in the proof of Theorem 1.3. Let use consider the case of $n=5$ and set

$$S=x^4+x^3y+x^2y^2+xy^3+y^4.$$

This can be written as

$$S=x^4+x^4\left(\frac{y}{x}\right)+x^4\left(\frac{y}{x}\right)^2+x^4\left(\frac{y}{x}\right)^3+x^4\left(\frac{y}{x}\right)^4$$

which is a geometric series. Thus

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{x^4 - x^4\left(\dfrac{y}{x}\right)^5}{1 - \dfrac{y}{x}} \\ &= \frac{x^5 - y^5}{x - y}. \end{aligned}$$

Recalling the definition of S this is

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5 - y^5}{x - y}$$

which is equivalent to the $n = 5$ version of Theorem 1.3. Give a proof of general case of Theorem 1.3 using the method just given. $\square$

Theorem 1.3 will be useful when we talk about difference quotients of polynomials (and more generally rational functions). We will be interested in simplifying expressions of the form

$$\frac{f(x) - f(a)}{x - a}$$

by trying to cancel the $(x - a)$ out of the denominator. This is becasue we will be wanting to compute limits

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

As an example let $f(x) = x^3 - 2x^2$. Then

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{x^3 - 2x^2 - (a^3 - 2a^2)}{x - a} \\ &= \frac{(x^3 - a^3) - 2(x^2 - a^2)}{(x - a)} \\ &= \frac{(x^3 - a^3) - 2(x^2 - a^2)}{x - a} \\ &= \frac{(x - a)(x^2 + xa + a^2) - 2(x - a)(x + a)}{x - a} \\ &= (x^2 + xa + a^2) - 2(x + a). \end{aligned}$$

and therefore

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a}((x^2 + xa + a^2) - 2(x + a)) = 3a^2 - 4a.$$

Problem 1.6. Let

$$f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

where c_0, c_1, c_2, and c_3 are constants. Simplify

$$\frac{f(x) - f(a)}{x - a}$$

by showing that $(x-a)$ can be canceled out of the denominator and use this to compute $\lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$. □

1.4. **Sums of arthritic series.** Anther sum that occurs naturally is

$$S = 1 + 2 + 3 + \cdots + n.$$

Let us compute this in some special cases. If

$$S = 1 + 2 + 3 + 4 + 5 + 6,$$

then also

$$S = 6 + 5 + 4 + 3 + 2 + 1.$$

We will add these together

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + 5 + 6 \\ S &= 6 + 5 + 4 + 3 + 2 + 1 \\ 2S &= (1 + 6) + (2 + 5) + (3 + 4) + (4 + 3) + (5 + 2) + (6 + 1) \\ &= 7 + 7 + 7 + 7 + 7 + 7 \qquad\qquad \text{(6 terms in the sum)} \\ &= 6 \cdot 7 \end{aligned}$$

Therefore

$$S = \frac{6 \cdot 7}{2} = 21.$$

This method works in general. If

$$S = 1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n$$

then we can reverse the sum

$$S = n + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1.$$

Adding these together gives

$$\begin{aligned} 2S &= (1 + n) + (2 + (n - 1)) + (3 + (n - 2)) + \cdots + ((n - 2) + 3) + ((n - 1) + 2) + (n + 1) \\ &= \underbrace{(n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1) + (n + 1) + (n + 1)}_{n \text{ terms}} \\ &= n(n + 1). \end{aligned}$$

Dividing by 2 gives

$$S = \frac{n(n + 1)}{2}.$$

This gives a proof of

Theorem 1.4. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

□

This can be generalized a bit. In general a finite ***arithmetic series*** is a sum of the form

$$\begin{aligned} S &= a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d) \\ &= \sum_{k=0}^{n-1}(a+kd). \end{aligned} \tag{1}$$

This sum has n terms. The number d is the ***common difference*** (or just the ***difference***) of the series.

Problem 1.7. (This problem as much about leaning to use summation notation as it is about the result.) Use summation notation and the generalization of the argument given here for $n = 5$ to derive a formula for the sum of the series (1). When $n = 5$ we have

$$S = a + (a+d) + (a+2d) + (a+3d) + (a+4d) = \sum_{k=0}^{4}(a+kd)$$

Writing this sum in the reverse order

$$S = (a+4d) + (a+3d) + (a+2d) + (a+d) + a = \sum_{k=0}^{4}\big(a+(4-k)d\big)$$

Therefore

$$\begin{aligned} 2S &= S + S \\ &= \sum_{k=0}^{4}(a+kd) + \sum_{k=0}^{4}\big(a+(4-k)d\big) \\ &= \sum_{k=0}^{4}\big((a+kd+a+(4-k)d\big) \\ &= \sum_{k=0}^{4}(2a+4d) \\ &= 5(2a+4d). \end{aligned}$$

Dividing by 2 then gives

$$S = 5(a+2d) = 5a + 10d.$$

Here is a way to rewrite this to make it seem more intuitive.

$$S = 5(a+2d) = 4\left(\frac{a+(a+4d)}{2}\right) = (\text{number of terms})\left(\frac{\text{first}+\text{last}}{2}\right)$$
$$= (\text{number of terms})\,(\text{average})$$

Thus the sum is the number of terms times the average of the first and last terms. Now you should do this argument in the case of general n. $\square$

1.5. **The binomial theorem.**

1.5.1. *Factorials and binomial coefficients.* We recall the definition of the ***factorials***. If n is a non-negative integer $n!$ is defined by

$$0! = 1 \quad \text{and for } n \geq 1 \quad n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

For small values of n we have

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880

n	$n!$
10	3,628,800
11	39,916,800
12	47,9001,600
13	622,7020,800
14	87,178,291,200
15	1,307,674,368,000
16	20,922,789,888,000
17	3556,87,428,096,000
18	6,402,373,705,728,000
19	121,645,100,408,832,000

n	$n!$
20	2,432,902,008,176,640,000
21	51,090,942,171,709,440,000
22	1,124,000,727,777,607,680,000
23	25,852,016,738,884,976,640,000
24	620,448,401,733,239,439,360,000
25	15,511,210,043,330,985,984,000,000
26	403,291,461,126,605,635,584,000,000
27	10,888,869,450,418,352,160,768,000,000
28	304,888,344,611,713,860,501,504,000,000
29	8,841,761,993,739,701,954,543,616,000,000
30	265,252,859,812,191,058,636,308,480,000,000

Problem 1.8. Show that for $n \geq 10$ that $n! \geq 3.6288(10)^{n-4}$. *Hint:* Use that $10! = 3{,}628{,}800 = 3.6288(10)^6$. For example if $n = 15$

$$\begin{aligned} 15! &= 10!(11)(12)(13)(14)(15) \\ &\geq 10!(10)(10)(10)(10)(10) \\ &= 10!(10)^5 \\ &= 3.6288(10)^6(10)^5 \\ &= 3.6288(10)^{11}. \end{aligned}$$

This idea works in general. □

Remark 1.5. These tables and the last problem make it clear that $n!$ grows very fast. There is a well known approximation, ***Stirling's formula***,

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

which shows that $n!$ grows faster than any exponential function. A more precise form of this was given by Herbert Robbins in 1955:

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}}.$$

for all positive integers n. Time permitting we will prove some form of Stirling's formula either this term or next term. □

An elementary property of factorials we will use many times is that we get $n!$ by multiplying $(n-1)!$ by n. Thus

$$\begin{aligned} n! &= n((n-1)!) \\ &= n(n-1)((n-2)!) \\ &= n(n-1)(n-2)((n-3)!) \end{aligned}$$

and so on. This especially useful when dealing with fractions involving factorials. For example:

$$\frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1)n((n-1)!)} = \frac{1}{(n+2)(n+1)n}.$$

Let $n, k \geq 0$ be integers with $0 \leq k \leq n$. Then the ***binomial coefficient*** $\binom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

This is read as "n choose k".

Problem 1.9. Show this this definition implies

$$\binom{n}{k} = \binom{n}{n-k}.$$

□

Also we generally do not have to compute $n!$ to find $\binom{n}{k}$ as lots of terms cancel. For example

$$\binom{100}{3} = \frac{100!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98 \cdot 97!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98}{3!} = 161{,}700.$$

Proposition 1.6. *The following hold*

$$\begin{aligned}
\binom{n}{0} &= \binom{n}{n} = 1,\\
\binom{n}{1} &= \binom{n}{n-1} = n,\\
\binom{n}{2} &= \binom{n}{n-2} = \frac{n(n-1)}{2},\\
\binom{n}{3} &= \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}.
\end{aligned}$$

Problem 1.10. Prove this. □

The expression $n(n-1)\cdots(n-k+1)$ comes up often enough that it is worth giving a name. Let $x^{\underline{k}}$ be the k***-th falling power*** of x. That is

$$x^{\underline{k}} := \begin{cases} 1, & k = 0 \\ x(x-1)\cdots(x-k+1), & k \geq 1. \end{cases}$$

Thus

$$\begin{aligned}
x^{\underline{0}} &= 1\\
x^{\underline{1}} &= x\\
x^{\underline{2}} &= x(x-1)\\
x^{\underline{3}} &= x(x-1)(x-2)\\
&\ \vdots\\
x^{\underline{k}} &= \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{k \text{ factors}}
\end{aligned}$$

Proposition 1.7. *The equality*

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

holds.

Problem 1.11. Prove this. □

Here is anther basic property of the binomial coefficients.

Proposition 1.8 (Pascal Identity)**.** *For $1 \leq k \leq n$ with k, n integers the equality*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

holds.

Problem 1.12. Prove this. *Hint:* Here is a special case

$$\begin{aligned}
\binom{12}{7} + \binom{12}{8} &= \frac{12!}{7!\,5!} + \frac{12!}{8!\,4!} \\
&= \frac{12!}{7!\,4!}\left(\frac{1}{5} + \frac{1}{8}\right) \\
&= \frac{12!}{7!\,4!}\left(\frac{8+5}{5\cdot 8}\right) \\
&= \frac{12!}{7!\,4!}\left(\frac{13}{5\cdot 8}\right) \\
&= \frac{13!}{8!\,5!} \\
&= \binom{13}{8}
\end{aligned}$$

where we have used $13! = 12!\cdot 13$, $8! = 7!\cdot 8$, and $5! = 4!\cdot 5$. □

If we put the binomial coefficients in a triangular table (***Pascal's triangle***):

$$\begin{array}{ccccccccccc}
 & & & & & \binom{1}{1} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ tells us that any entry is the sum of the two entries directly above. This can be used to compute $\binom{n}{k}$ for small values of n. For example up to $n = 5$ the binomial coefficients are given by:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The following problem is both interesting in its own right and is a chance to review induction.

Problem 1.13. Let k, n be nonnegative integers with $0 \le k \le n$. Prove the binomial coefficient $\binom{n}{k}$ is an integer. *Hint:* We use induction on n. We can either use $n = 0$ or $n = 1$ as the base case as

$$\binom{0}{0} = 1, \text{ and } \binom{1}{0} = \binom{1}{1} = 1$$

and 1 is an integer. We now use induction on n. Assume that

$$\binom{n}{k} \text{is an integer for } 0 \le k \le n. \tag{2}$$

To complete the induction step we need to show

$$\binom{n+1}{k} \text{is an integer for } 0 \le k \le n+1. \tag{3}$$

This is true for the values $k = 0$ and $k = n + 1$ as

$$\binom{n+1}{0} = \binom{n+1}{n+1} = 1.$$

Thus we can assume $1 \le k \le n$. By the Pascal Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

and now use (2) to show that (3) holds to complete the induction. $\square$

1.5.2. *The binomial theorem.* One reason the binomial coefficients are important is

Theorem 1.9 (Binormal Theorem)**.** *For any positive integer n and $x, y \in \mathbb{R}$*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

In summation notation this is

$$(x + y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We will prove this shortly. For $n = 5$ we have

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Let $x = y = 1$ in this to get

$$\begin{aligned} 2^5 &= (1 + 1)^5 \\ &= (1)^5 + 5(1)^4(1) + 10(1)^3(1)^2 + 10(1)^2(1)^3 + 5(1)(1)^4 + (1)^5 \\ &= 1 + 5 + 10 + 10 + 5 + 1, \end{aligned}$$

which may not be that interesting of a fact, but the argument lets us see a pattern for something that is interesting.

Problem 1.14. Use this idea to show the sum of the numbers $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ is 2^n. That is for all positive integers n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

□

Problem 1.15. Prove for any positive integer n that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: $(1-1) = 0$. □

Here is a bit of practice in using the binomial theorem.

Problem 1.16. Expand the following:

(a) $(1 + 2x^3)^4$,
(b) $(x^2 - y^5)^3$.

Problem 1.17. Use induction and the Pascal Identity to prove the Binomial Theorem. *Hint:* Use for the base case that $(x + y)^1 = x + y$. Here is what the induction step from $n = 4$ to $n = 5$ looks like. Assume that we know that

$$\begin{aligned}(x+y)^4 &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3 + \binom{4}{4} y^4 \\ &= x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3 + y^4\end{aligned}$$

where we have used that $\binom{4}{0} = \binom{4}{4} = 1$. We now want to show the theorem holds for $n = 5$.

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= (x+y)\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&\quad + y\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x^5 + \binom{4}{1}x^4y + \binom{4}{2}x^3y^2 + \binom{4}{3}x^2y^3 + xy^4 \\
&\quad + x^4y + \binom{4}{1}x^3y^2 + \binom{4}{2}x^2y^3 + \binom{4}{3}x^1y^4 + y^5 \\
&= x^5 + \left(\binom{4}{0} + \binom{4}{1}\right)x^4y + \left(\binom{4}{1} + \binom{4}{2}\right)x^3y^2 \\
&\quad + \left(\binom{4}{2} + \binom{4}{3}\right)x^2y^3 + \left(\binom{4}{3} + \binom{4}{4}\right)xy^4 + y^5 \\
&= x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5
\end{aligned}$$

If you don't like this long hand what of doing it, here is what the same calculation looks like using summation notation. Assume that

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k}.$$

Then

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= x(x+y)^4 + y(x+y)^4 \\
&= x\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} + y\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} \\
&= \sum_{k=0}^{4}\binom{4}{k}x^{k+1} y^{4-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^{k} y^{4-(k-1)} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^{k} y^{5-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \binom{4}{4}x^5 + \binom{4}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1}+\binom{4}{k}\right)x^k y^{5-k} \\
&= \binom{5}{5}x^5 + \binom{5}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1}+\binom{4}{k}\right)x^k y^{5-k} \\
&= \binom{5}{5}x^5 + \binom{5}{0}y^5 + \sum_{k=1}^{4}\binom{5}{k}x^k y^{5-k} \\
&= \sum_{k=0}^{5}\binom{5}{k}x^k y^{5-k}.
\end{aligned}$$

where we have done the change of variable $k \mapsto k-1$ in the first sum on line 5, used the Pascal Identity to get to the second to the last line, and used that $\binom{4}{0} = \binom{5}{0} = 1$ and $\binom{4}{4} = \binom{5}{5} = 1$.

Either of these two calculations shows that if the Binomial Theorem holds for $n = 4$ then it holds for $n = 5$. Use a similar calculation to show that if the theorem holds for n, then in holds for $n+1$. □

Here is an example of one (of the many) ways we will be using the binomial theorem. Similar to some examples given above we will want to simplify expressions of the form

$$\frac{f(x+h)-f(x)}{h}$$

by cancelling the h out of the denominator so that we can compute the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Here is an example. Let $f(x) = x^4$. Then

$$\begin{aligned}\frac{f(x+a)-f(x)}{h} &= \frac{(x+h)^4 - y^4}{h}\\ &= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}\\ &= \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}\\ &= 4x^3 + 6x^2h + 4xh^2 + h^3.\end{aligned}$$

Whence

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3.$$

1.6. **Appendix: Elements of the theory of finite differences.** Let $f\colon \mathbb{Z}\to\mathbb{R}$ be a function from the integers, $\mathbb{Z}$, to the real numbers, $\mathbb{R}$. We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

For example we will be able to show

$$\begin{aligned}1 + 2 + \cdots + n &= \frac{n(n+1)}{2}\\ 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6}\\ 1^3 + 2^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4}.\end{aligned}$$

1.6.1. *The difference operator and the Fundamental Theorem of Summation Theory.*

Definition 1.10. Let $f\colon \mathbb{Z}\to\mathbb{R}$. Then the ***diffenence***, Δf, of f is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator Δ is called the ***difference operator***. □

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1)+2) - (3x+2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table a, b, c, r are constants.

$f(x)$	$\Delta f(x)$
c	0
$ax+b$	a
ar^x	$a(r-1)r^x$

Problem 1.18. Verify these. □

Theorem 1.11 (**Fundamental Theorem of Summation Theory).** *Let $f\colon \mathbb{Z}\to\mathbb{R}$ and let F be an **anti-difference** of f. That is $\Delta F = f$. Then for $a, b\in\mathbb{Z}$ with $a<b$*

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

In particular

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1).$$

Proof. This uses the basic trick about telescoping sums:

$$\begin{aligned}\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b}(F(k+1)-F(k))\\ &= \sum_{k=a}^{b} F(k+1) - \sum_{k=a}^{b} F(k)\\ &= \big(F(a+1)+F(a+2)+\cdots+F(b)+F(b+1)\big)\\ &\qquad - \big(F(a)+F(a+1)+\cdots+F(b-1)+F(b)\big)\\ &= F(b+1)-F(a)\end{aligned}$$

as required. □

Theorem 1.11 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences (a, r and b are constants).

$f(x)$	$F(x)$
$ax+b$	$a\dfrac{x(x-1)}{2}+bx$
ar^x	$\dfrac{ar^x}{r-1}$

Problem 1.19. Verify these. (You just need to check $F(x+1)-F(x) = f(x)$). □

Problem 1.20. Use that $\dfrac{ar^x}{r-1}$ is the anti-difference of ar^x and Theorem 1.11 give anther proof of

$$a + ar + ar^2 + \cdots + ar^n = \frac{a-ar^{n+1}}{1-r} = \frac{\text{first} - \text{next}}{1-\text{ratio}}.$$

□

1.6.2. *Falling factorial powers and sums of powers.* For Theorem 1.11 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

Definition 1.12. For natural number p define the ***falling factorial power*** of $x \in \mathbb{R}$ as $x^{\underline{0}} = 1$ and for $p \geq 1$

$$x^{\underline{p}} = x(x-1)(x-2)\cdots(x-(p-1)).$$

(This product has p terms.) □

For small values of p this becomes

$$\begin{aligned}
x^{\underline{0}} &= 1\\
x^{\underline{1}} &= x\\
x^{\underline{2}} &= x(x-1)\\
x^{\underline{3}} &= x(x-1)(x-2)\\
x^{\underline{4}} &= x(x-1)(x-2)(x-3)\\
x^{\underline{5}} &= x(x-1)(x-2)(x-3)(x-4).
\end{aligned}$$

Proposition 1.13. *If $f(x) = x^{\underline{p}}$ where p is a natural number, then $\Delta f(x) = px^{\underline{p-1}}$. That is*

$$\Delta x^{\underline{p}} = px^{\underline{p-1}}.$$

Problem 1.21. Prove this. *Hint:* Here is what the calculation looks like when $p = 5$.

$$\begin{aligned}
\Delta x^{\underline{5}} &= (x+1)^{\underline{5}} - x^{\underline{5}}\\
&= (x+1)x(x-1)(x-2)(x-3) - x(x-1)(x-2)(x-3)(x-4)\\
&= \big((x+1)-(x-4)\big)x(x-1)(x-2)(x-3)\\
&= 5x^{\underline{4}}.
\end{aligned}$$

□

Remark 1.14. The formula should remind you of the formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. □

Proposition 1.15. *If $f(x) = x^{\underline{p}}$ where p is a non-negative integer, then $F(x) = \dfrac{1}{p+1}x^{\underline{p+1}}$ is an anti-difference of f.*

Problem 1.22. Prove this as a corollary of Proposition 1.13 by noting (by replacing p by $p+1$), that $\Delta x^{\underline{p+1}} = (p+1)x^{\underline{p}}$ and dividing by $(p+1)$. □

Problem 1.23. Show that if $p \geq 2$ that $1^{\underline{p}} = 0$. (For example $1^{\underline{3}} = 1(1-1)(1-2) = 0$.) □

Proposition 1.16. *If p is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

Remark 1.17. This should remind you of the formula $\displaystyle\int_0^x t^p\,dt = \frac{x^{p+1}}{p+1}$.

Problem 1.24. Prove this. HINT: Let $f(x) = x^{\underline{p}}$. Then $F(x) = \dfrac{x^{\underline{p+1}}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 1.11

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

and use Problem 1.23 to see that $F(1) = 0$. □

Proposition 1.18. *The equalities*

$$\begin{aligned}
x &= x^{\underline{1}} \\
x^2 &= x^{\underline{2}} + x^{\underline{1}} \\
x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\
x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\
x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}}
\end{aligned}$$

hold.

Problem 1.25. Verify the first three of these. □

Problem 1.26. Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3.$$

HINT: Here is the idea for $\sum_{k=1}^{n} k^2$. Using the last problem and Proposition 1.16

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) \\
&= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}.
\end{aligned}$$

We can leave the answer like this, or expand and factor to get
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$
Do similar calculations for $\sum_{k=1}^{n} k^3$. □

1.6.3. *A couple of trigonometric sums.* For your convenience we recall some trig identities:
$$\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\\
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)
\end{aligned}$$

Problem 1.27. Let θ be a constant with $\sin(\frac{\theta}{2}) \neq 0$. Use the identities above to show
$$\sin\left(\theta\left(x+\frac{1}{2}\right)\right) - \sin\left(\theta\left(x-\frac{1}{2}\right)\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\theta x\right)$$
and therefore
$$F(x) = \frac{\sin\left(\theta\left(x-\frac{1}{2}\right)\right)}{2\sin(\frac{\theta}{2})}.$$
is an anti-difference of
$$f(x) = \cos(\theta x).$$
□

Proposition 1.19. *If* $\sin\left(\frac{\theta}{2}\right) \neq 0$, *then*
$$\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}.$$

Problem 1.28. Use Problem 1.27 and Theorem 1.11 to prove this. □

There is a similar formula for sums for the sine function.

Proposition 1.20. *If* $\sin(\frac{\theta}{2}) \neq 0$, *then*
$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}.$$

Problem 1.29. Prove this. □

2. The real numbers.

Our short term goal is to give a precise description of the real numbers, $\mathbb{R}$. This will involve three aspects of them. The first is the usual algebraic properties (addition, multiplication, etc.), order properties (the basic properties of inequalities) and finally a completeness property (the least upper bound axiom) which in one form says that there are no "holes" in the real numbers.

2.1. **Fields.** Here we deal with the algebraic properties of the real numbers. A ***field*** is an algebraic object where we can do the usual operations of high school algebra. That is addition, subtraction, multiplication, and division.

Definition 2.1. A ***field*** is a set F with operations[2] $+$, called ***addition***, and $\cdot$, called ***multiplication***, such that

(a) both operations are ***associative***:
$$(x+y)+z = x+(y+z) \qquad x\cdot(y\cdot z) = (x\cdot y)\cdot z$$
for all $x, y, z \in F$.

(b) both operations are ***commutative***:
$$x+y = y+x \qquad x\cdot y = y\cdot x$$
for all $x, y \in F$.

(c) Multiplication ***distributes over addition***:
$$x\cdot(y+z) = x\cdot y + x\cdot z.$$
for all $x, y, z \in F$.

(d) There are additive and multiplicative ***identities***. That is there are $0 \in F$ and $1 \in F$ such that
$$x+0 = x \qquad x\cdot 1 = x$$
for all $x \in F$.

(e) Every element has an ***additive inverse***. That is for every $x \in F$ there is an element $y \in F$ such that
$$x+y = 0.$$

(f) Every nonzero element has a ***multiplicative inverse***. There is for $x \in F$, with $x \neq 0$, there is an element $z \in F$ such that
$$xz = 1.$$

(g) F has at least two elements.

This definition requires a bit of comment. First as to the additive identity the definition as it stands does not rule out the possibility that there are two additive identities. that is there are $0, 0' \in F$ with
$$x+0 = x \quad \text{and} \quad x+0' = x$$
for all $x \in F$. In this case
$$\begin{aligned} 0' &= 0' + 0 && (x+0 = x \text{ with } x = 0') \\ &= 0 + 0' && (\text{addition is commutative}) \\ &= 0 && (x + 0' = x \text{ with } x = 0). \end{aligned}$$
So 0 and $0'$ are the same element.

[2]To be a bit more precise we should call these ***binary operations*** in that that they take an ordered pair of elements of F, say (x, y), and each gives a unique output $x+y$ or $x\cdot y$.

Problem 2.1. Use a variant on this argument to show that if $1, 1' \in F$ satisfy

$$x \cdot 1 = x \quad \text{and} \quad x \cdot 1' = x$$

for all $x \in F$ that $1 = 1'$. Thus the multiplicative identity is unique. $\square$

We also have that additive inverses are unique. Let $x \in F$ and assume that $y, y' \in F$ such that

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

Then

$$\begin{aligned}
y' &= y' + 0 && (x + 0 = x \text{ with } x = y') \\
&= y' + (x + y) && (x + y = 0) \\
&= (y' + x) + y && (\text{associativity of addition}) \\
&= (x + y') + y && (\text{commutative of addition}) \\
&= 0 + y && (x + y' = 0) \\
&= y + 0 && (\text{commutative of addition}) \\
&= y && (x + 0 = x \text{ with } x = y)
\end{aligned}$$

Thus the additive inverse of any element any element is unique. From now on we denote the additive inverse of $x \in F$ as $-x$ and use the abbreviation

$$x - y := x + (-y).$$

Proposition 2.2. *For any $x \in F$ the equality*

$$-(-x) = x$$

holds.

Proof. By definition $-(-x)$ is the additive inverse of $-x$. But we also have

$$\begin{aligned}
-x + x &= x + (-x) && (\text{commutative of additive}) \\
&= 0 && (-x \text{ is additive inverse of } x)
\end{aligned}$$

This shows that x is also an additive inverse of $-x$. As additive inverses are unique we have $-(-x) = x$. $\square$

Problem 2.2. Modify the argument above to show that the multiplicative inverse of $x \in F$ with $x \neq 0$ is unique. $\square$

If $x \in F$ and $x \neq 0$ we now denote the unique multiplicative inverse of x by either of the two notations.

$$\text{multiplicative inverse of } x = \frac{1}{x} = x^{-1}.$$

and write

$$yx^{-1} := \frac{y}{x}.$$

Problem 2.3. Modify one of the arguments above to show if $x \in F$ with $x \neq 0$ then

$$(x^{-1})^{-1} = x.$$

That is

$$\frac{1}{\left(\frac{1}{x}\right)} = x.$$

□

Here are several results that we are so use to seeing that it seems irritating to have to prove them.

Proposition 2.3. *In a field* $-0 = 0$.

Problem 2.4. Prove this. *Hint:* $0 + 0 = 0$ so 0 is the additive inverse of 0. □

Problem 2.5. In a field, F,

$$x \cdot 0 = 0$$

for all $x \in F$.

Problem 2.6. Prove this. *Hint:* First show $x \cdot 0 = x \cdot 0 + x \cdot 0$ by justifying the steps in the following.

$$\begin{aligned} x \cdot 0 &= x \cdot (0 + 0) \\ &= x \cdot 0 + x \cdot 0. \end{aligned}$$

Now add the additive inverse of $x \cdot 0$ to both sides of $x \cdot 0 = x \cdot 0 + x \cdot 0$. □

The associativity law implies that for any three elements $x_1, x_2, x_3 \in F$ that

$$(x_1x_2)x_3 = x_1(x_2x_3).$$

As this is the only two ways to group the product of three elements we can write the product of three elements as

$$x_1x_2x_3$$

without ambiguity. There are five ways to group four elements in a product

$$x_1(x_2(x_3x_4)),\ x_1((x_2x_3)x_4),\ (x_1x_2)(x_3x_4),\ (x_1(x_2x_2))x_4,\ ((x_1x_2)x_3)x_4$$

These are all equivalent. We see this by showing they are all the same as $x_1(x_2(x_3x_4))$.

$$\begin{aligned} x_1((x_2x_3)x_4) &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_2x_3)x_4 = x_2(x_3x_4) \\ (x_1x_2)(x_3x_4) &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4 \\ (x_1(x_2x_3))x_4 &= x_1((x_2x_3)x_4) && \text{as} \quad (x_1y)x_4 = x_1(yx_4) \text{ with } y = x_2x_3 \\ &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_2x_3)x_4 = x_2(x_3x_4) \\ ((x_1x_2)x_3)x_4 &= (x_1x_2)(x_3x_4) && \text{as} \quad (yx_3)x_4 = y(x_3x_4) \text{ with } y = x_1x_2 \\ &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4 \end{aligned}$$

So again we can write the product

$$x_1x_2x_3x_4$$

without ambiguity as all the groupings are equal. In light of this the following will most likely not surprise you.

Proposition 2.4. *Let $x_1x_2, \ldots, x_n$ be elements of the field. Then the associativity law implies that any two groupings of the product $x_1x_2\cdots x_n$ are equal.*

Problem 2.7. Prove this. *Hint:* Use induction to show that any grouping is equal to the grouping

$$x_1(x_2(x_3(x_4\cdots x_n)\cdots)).$$

This is the grouping where the parenthesis are moved as far to the right as possible. For the rest of this problem call this the ***standard form*** of the product.

Here is the induction step in going from $n = 5$ to $n = 6$. For $n = 5$ the standard form is

$$x_1(x_2(x_3(x_4x_5))).$$

Let p be some grouping of $x_1, x_2, \ldots, x_6$. We first consider the case that p is of the form

$$p = x_1(p_2)$$

where p_2 is a product of $x_2, \ldots, x_6$. Then p_2 is a product of $n = 5$ elements and thus by the induction hypothesis $p_2 = x_2(x_3(x_4(x_5x_6)))$. But then $p = x_1(p_2) = x_1(x_2(x_3(x_4(x_5x_6))))$ can be put in standard form.

This leaves the case where $p = (p_1)(p_2)$ where for some k with $2 \leq k \leq 5$ we have

$$p = (p_1)p_2$$

where p_1 is a product of $x_1, \ldots, x_k$ and p_2 is a product of $x_{k+1}, \ldots, x_6$. Then, as p_1 has less than $n = 6$ factors it can be put in standard form. This implies that $p_1 = x_1(q)$ where q is a product of $x_2, \ldots, x_k$. Therefore

$$p = (p_1)p_2 = (x_1q)p_2 = x_1(qp_2).$$

But qp_2 only involves the variables $x_2, \ldots, x_6$ so anther application of the induction hypothesis implies that qp_2 can be put standard form. But then $p = x_1(qp_2)$ is in standard form.

To complete the proof you show show that this argument can be used to show that if it is true for n variables, then it is true of $n + 1$ variables. $\square$

From now on we write products $x_1x_2\cdots x_n$ without putting the the parenthesis. There is a similar proposition about parenthesis and and sums, and we will write also write sums as $x_1 + x_2 + \cdots + x_n$ without parenthesis.

The following could be summarized by saying that much of the basic results you know from basic algebra still holds in fields.

Proposition 2.5. *Let F be a field. Then* *(a) If* $a, b, c, d \in F$ *and* $b, c \neq 0$ *then*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

(b) If $a, b \in F$ *and* $a, b = 0$, *then* $a = 0$ *or* $b = 0$.

(c) (This is just a useful restatement of part (b).) If $a, b \in F$ *and* $a, b \neq 0$ *then* $ab \neq 0$.

(d) If the elements $a_1, a_2, \ldots, a_n$ F *are all nonzero, then so is the product and*

$$(a_1 a_2 \cdots a_{n-1} a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_2^{-1} a_1^{-1}.$$

(e) If $a, b \in F$ *and* $a^2 = b^2$, *then* $a = \pm b$.

Problem 2.8. Prove this. □

Problem 2.9 (Cramer's rule for solving linear systems.). Here is anther fact that will likely come up at least one during the term. Let a, b, c, d, e, f be elements of the field F with

$$ad - bc \neq 0.$$

Then the equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

have a unique solution. This solution is

$$x = \frac{ed - bf}{ad - bc}, \qquad y = \frac{af - ec}{ad - bd}.$$

Hint: To find x multiply the first equation by d and the second by b and then subtract the two. A similar trick works to find y. □

2.1.1. *Some examples of fields. The rational numbers.*

We first recall some sets of numbers that occur often enough that they have earned names. First there is the set of ***natural numbers***,

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

and the set of ***integers***

$$\mathbb{Z} = \{\ldots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

Thus the natural numbers are just the positive integers.[3] Then the ***rational numbers*** are

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

That is rational numbers are the quotients of integers where the denominators are not zero. It is not hard to check that $\mathbb{Q}$ is a field.

[3]Some books and some mathematicians include 0 in the set of natural numbers, so that for them $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$. I am going to use the same convention our text uses.

The rational numbers are fine for doing some parts of algebra, for example if $a, b \in \mathbb{Q}$ with $a \neq 0$ we can always solve the equation

$$ax + b = 0$$

for x. More generally if $a, b, c, d, e, f \in \mathbb{Q}$ and $ad - bc \neq 0$ then we can solve

$$ax + by = e$$
$$cx + dy = f$$

for x and y and get rational numbers as solutions. In your linear algebra class you saw that in any field that you can solve consistent systems of linear equations of any size and get solutions that are in the same field as the coefficients.

However there are natural equations that do not have solutions in the rational number. For example $x^2 - 2 = 0$ has no rational solution as $\sqrt{2}$ is irrational. More generally we have:

Theorem 2.6. *If m is a positive integer that is not a perfect square (that is $m \neq k^2$ for any integer k) then the equation*

$$x^2 = m$$

has no solution in the rational numbers. (That is $\sqrt{m}$ is irrational.)

Proof. Towards a contradiction assume $x = \sqrt{m}$ is a rational number that is not an integer. Let n be the smallest positive integer such that the product nx is an integer.

Let $\lfloor x \rfloor$ be the greatest integer in x. Then $0 \leq x - \lfloor x \rfloor < 1$. And as x is not an integer $x \neq \lfloor x \rfloor$ and so $0 < x - \lfloor x \rfloor < 1$. Let $p = n(x - \lfloor x \rfloor)$. Then $0 < p < n$ and $p = nx - n\lfloor x \rfloor$ is an integer. But, using that $x^2 = m$,

$$px = n(x - \lfloor x \rfloor)x = nx^2 - (nx)\lfloor x \rfloor) = nm - (nx)\lfloor x \rfloor$$

which is an integer. As $p < n$ this contradicts that n was the smallest positive integer such that the product nx is an integer. $\square$

The last theorem shows that all the square roots that you expect to be irrational ($\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, etc.) are irrational. In some vague sense this tells use that the rational numbers are not "complete" in the sense that there is a missing number or a "hole" where $\sqrt{2}$ should be.

2.1.2. *An aside: Some other examples of fields.* While these will not come up again in this course, it is interesting to see a couple of other examples of fields.

The first example is the integers modulo a prime, p. One description of this is the numbers $0, 1, \ldots, (p-1)$ and then when adding or multiplying them reduce them modulo p. (If n is an integer, then n reduced mod p is the remainder when p is divided into n.) For $p = 5$ here are what the addition and multiplication tables.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

+	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Thus there are fields that only have a finite number of elements.

The second example is the rational numbers extended by and irrational square root of an integer. Let m be a positive integer that is not a perfect square. Then by Theorem 2.6 the number $\sqrt{m}$ is irrational. Let

$$\mathbb{Q}(\sqrt{m}) := \{a + b\sqrt{m} : a, b \in \mathbb{Q}\}.$$

This set is closed under addition and subtraction. Also as

$$(a + b\sqrt{m})(c + d\sqrt{m}) = (ac + bdm) + (ad + bc)\sqrt{m}$$

we see that $\mathbb{Q}(\sqrt{m})$ is closed under multiplication. For it to be a field still need that multiplicative inverses exist.

$$\begin{aligned}
\frac{1}{a + b\sqrt{m}} &= \frac{a - b\sqrt{m}}{(a + b\sqrt{m})(a - b\sqrt{m})} \\
&= \frac{a - b\sqrt{m}}{a^2 - b^2 m} \\
&= \frac{a}{a^2 - b^2 m} + \frac{-b}{a^2 - b^2 m}\sqrt{m} \\
&= \alpha + \beta\sqrt{m}
\end{aligned}$$

where

$$\alpha = \frac{a}{a^2 - b^2 m}, \qquad \beta = \frac{-b}{a^2 - b^2 m}$$

are rational numbers, provided the denominator $a^2 - b^2 m \neq 0$. But if $a^2 - b^2 m = 0$, then $m = (a/b)^2$, contradicting that $\sqrt{m}$ is irrational. Thus we can add, subtract, multiply, and divide in $\mathbb{Q}(\sqrt{m})$ and so $\mathbb{Q}(\sqrt{m})$ is a field. These fields are useful in number theory.

2.2. **The order axioms.** Let $\mathbb{F}$ be a field. Then an ***ordering*** of $\mathbb{F}$ is a subset $\mathbb{F}_+$, called the set of ***positive elements***, such that the following hold.

Pos 1: The set of positive elements is closed under addition and multiplication. That is if $a, b \in \mathbb{F}_+$ then $a + b \in \mathbb{F}_+$ and $ab \in \mathbb{F}_+$.

Pos 2: (The trichotomy principle.) For any $a \in \mathbb{F}$ exactly one of the following holds:

$$\begin{aligned}
&a \in \mathbb{F}_+ \\
&a = 0 \\
&-a \in \mathbb{F}_+.
\end{aligned}$$

We define the set ***negative elements*** of $\mathbb{F}$ as

$$\mathbb{F}_- = \{a : -a \in \mathbb{F}_+\}.$$

Now define the relation $<$ on $\mathbb{F}$ by

$$a < b \quad \text{if and only if} \quad b - a \in \mathbb{F}_+.$$

(Likewise $b < a$ is defined by $b - a \in \mathbb{F}_+$, which is the same as $a > b$.) We define $a \leq b$ by

$$a \leq b \quad \text{if and only if} \quad a = b \text{ or } a < b.$$

with a similar definition for $a \geq b$. As expected $a <$ is read as "a is less than b" and $a \leq b$ is real as "a is less than or equal to b".

With this definition and terminology we have that the set of positive elements is the same as the set of elements that are greater than 0.

Proposition 2.7 (Trichotomy for inequalities.). *If $a, b \in \mathbb{F}$, then exactly one of the following holds:*

$$a < b, \quad a = b, \quad a > b.$$

Problem 2.10. Prove this. *Hint:* Apply **Pos: 1** to $b - a$.

Proposition 2.8 (Transitivity). *If $a < b$ and $b < c$ then $a < c$.*

Problem 2.11. Prove this. □

Proposition 2.9. *If $a < b$ and $c < d$ then $a + c < b + d$.*

Problem 2.12. Prove this. □

Proposition 2.10. *If $a < b$ and $c > 0$, then $ac < bc$.*

Problem 2.13. Prove this. □

Proposition 2.11. *If $a < b$ and $c < 0$, then $ac > bc$.*

Problem 2.14. Prove this. □

Proposition 2.12. *If $0 < a < b$ and $0 < c \leq d$, then $ac < bd$.*

Problem 2.15. Prove this. □

Proposition 2.13. *If $a_1, a_2, \ldots, a_n > 0$ then $a_1 a_2 \cdots a_n > 0$ and and $a_1 + a_2 + \cdots + a_n > 0$. (That is the sum and product of positive numbers is positive.) Thus if $a > 0$ then a^n and $na > 0$.*

Problem 2.16. Prove this. *Hint:* Use this problem to practice using induction. □

Proposition 2.14. *If $a \neq 0$, then $a^2 > 0$. That is the square of any nonzero element is positive. In particular $1 = 1^2$ is positive.*

Problem 2.17. Prove this. *Hint:* By trichotomy we have $a > 0$, $a = 0$, or $a < 0$. We are assuming that $a \neq 0$. So this leaves two cases: If $a > 0$, then as the positive numbers are closed under multiplication $a^2 = aa > 0$. If $-a > 0$ then $a^2 = (-a)^2 = (-a)(-a) > 0$. □

Proposition 2.15. *Let $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Then*

$$a_1^2 + a^2 + \cdots + a_n \geq 0$$

with equality if and only if $a_1 = a_2 = \cdots = a_n = 0$. That is the sum of squares of elements from $\mathbb{F}$ is positive unless all the elements are zero.

Problem 2.18. Prove this. *Hint:* Use induction. □

Proposition 2.16. *If $a > 0$, then $1/a > 0$. If $a < 0$ then $1/a < 0$.*

Problem 2.19. Prove this. *Hint:* Towards a contradiction, assume that $a > 0$ and $1/a < 0$. Then use $1 = a(1/a)$ to get the contradiction. □

Proposition 2.17. *If $0 < a < b$, then $1/b < 1/a$.*

Problem 2.20. Prove this. *Hint:* Multiply the inequality $a < b$ by the positive number $1/ab$. □

If $a \in \mathbb{F}$ we define the ***absolute value*** of a by

$$|a| = \begin{cases} a, & a > 0; \\ 0, & a = 0; \\ -a, & a < 0. \end{cases}$$

One way to think of $|a|$ is that it is the distance of a from the origin. Inequalities involving absolute values will come up repeatedly in what follows. Here are some of the basic ones. Note that a direct consequence of the defintion is that

$$|-a| = |a|.$$

Proposition 2.18. *For $a \in \mathbb{F}$,*

$$|a| \geq 0$$

with equality if and only if $a = 0$.

Problem 2.21. Prove this. *Hint:* Consider the three case $a > 0$, $a = 0$, and $a < 0$. □

Proposition 2.19. *For $a \in \mathbb{F}$ we have $a \leq |a|$.*

Problem 2.22. Prove this. □

Proposition 2.20. *For $a \in \mathbb{F}$ we have $a^2 = |a|^2$.*

Problem 2.23. Prove this. □

Proposition 2.21. *If $a, b \in \mathbb{F}$, then the following are equivalent:*
(a) $|a| = |b|$,
(b) $a = \pm b$,
(c) $a^2 = b^2$.

Problem 2.24. Prove this. □

The following can be useful in proving inequalities about absolute values:

Proposition 2.22. *If $a, b \in \mathbb{F}$,*

$$|a| < |b| \quad \text{if and only if} \quad a^2 < b^2$$

Proof. First assume that $|a| < |b|$. Then $|b| - |a| > 0$. Multiply this inequality by the positive element $|b| + |a|$ to get

$$(|b| + |a|)(|b| - |a|) > 0.$$

This simplifies to

$$|b|^2 - |a|^2 > 0.$$

But $|a|^2 = a^2$ and $|b|^2 = b^2$ and thus

$$b^2 - a^2 > 0$$

which implies $a^2 < b^2$. This shows $|a| < |b|$ implies $a^2 < b^2$.

Conversely assume $a^2 < b^2$. Then $0 < b^2 - a^2$. Again using that $|a|^2 = a^2$ and $|b|^2 = b^2$ we have

$$0 < b^2 - a^2 = |b|^2 - |a|^2 = \big(|b| + |a|\big)\big(|b| - |a|\big).$$

Multiply this inequality by the positive element $1/(|b| + |a|)$ to get

$$|b| - |a| > 0$$

which implies $|a| < |b|$. Thus we have shown $a^2 < b^2$ implies $|a| < |b|$, which completes the proof. $\square$

Here is a slight variant on the last proposition.

Proposition 2.23. *If $a, b \in \mathbb{F}$, then*

$$|a| \leq |b| \quad \text{if and only if} \quad a^2 \leq b^2.$$

Proof. This follows from the last proposition by splitting into the two cases $|a| = |b|$ and $|a| < |b|$. $\square$

Proposition 2.24. *Let $a > 0$. Then for $x \in \mathbb{F}$*

$$|x| < a \quad \text{if and only if} \quad -a < x < a.$$

Likewise

$$|x| \leq a \quad \text{if and only if} \quad -a \leq x \leq a.$$

Problem 2.25. Prove this. $\square$

Proposition 2.25 (The triangle inequality)**.** *If $a, b \in \mathbb{F}$, then*

$$|a + b| \leq |a| + |b|.$$

Proof. The most straightforward proof of this is by considering cases. But the number of cases is large enough that this is no fun. Here is a somewhat more insightful method idea.

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 && (\text{as } a^2 = |a|^2 \text{ and } b^2 = |b|^2) \\ &\le |a|^2 + 2|a||b| + |b|^2 && (\text{as } ab \le |ab| = |a||b|) \\ &\le (|a| + |b|)^2\end{aligned}$$

That is $(a+b)^2 \le (|a|+|b|)^2$. Therefore by Proposition 2.23

$$|a+b| \le \big||a|+|b|\big| = |a|+|b|$$

where $\big||a|+|b|\big| = |a|+|b|$ because $|a|+|b|$ is positive. □

Proposition 2.26 (The reverse triangle inequality.)**.** *If $a, b \in \mathbb{F}$, then*

$$\big||a|-|b|\big| \le |a-b|.$$

Understanding the absolute value of products and quotients is easier:

Proposition 2.27. *If $a, b \in \mathbb{F}$, then*

$$|ab| = |a||b|$$

and if $b \neq 0$,

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

Problem 2.26*.* Prove this. □

We will use the usual notation to define intervals in $\mathbb{F}$. That is

$$\begin{aligned}(a,b) &= \{x \in \mathbb{F} : a < x < b\} \\ [a,b) &= \{x \in \mathbb{F} : a \le x < b\} \\ (a,b] &= \{x \in \mathbb{F} : a < x \le b\} \\ [a,b] &= \{x \in \mathbb{F} : a \le x \le b\}\end{aligned}$$

where $a, b \in \mathbb{F}$ and $a < b$. The length of any of these intervals is $b - a$.

For any $x_1, x_2 \in \mathbb{F}$, we can think of $|x_2 - x_1|$ as the distance between x_1 and x_2. If $x_1, x_2 \in [a,b]$ then the distance between x_1 and x_2 should be at most the length of the interval. The following makes this precise.

Proposition 2.28. *If $a < b$ and $x_1, x_2 \in [a,b]$, then*

$$|x_2 - x_1| \le (b-a).$$

Likewise if $a < b$ and $(x_1, x_2) \in (a,b)$, then

$$|x_2 - x_1| < (b-a).$$

Problem 2.27*.* Prove this. □

2.2.1. *Some practice with inequalities.* Being able to work with inequalities is a one of the main tools in analysis. In this subsection we give some examples and problems using inequalities. We will be using the following generalization of the triangle inequality:

Proposition 2.29. *If $a_1, a_2, \dots, a_n \in \mathbb{F}$, then*

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots |a_n|.$$

Proof. This is an easy induction proof and is left to the reader. □

Example 2.30. Let $\delta > 0$ and let $|x - a| < \delta$. Then

$$|(3x + 2) - (3a + 2)| < 3\delta.$$

Solution: This one is just a matter of regrouping:

$$\begin{aligned} |(3x + 2) - (3a + 2)| &= |3(x - a)| \\ &= 3|x - a| \\ &< 3\delta. \end{aligned}$$

□

Problem 2.28. Let $\delta > 0$. Assume that $|y - c| < \delta$. Show that

$$|(-5y + 42) - (-5c + 42)| < 5\delta.$$

Example 2.31. If $|x - a| < \delta \le 1$, then

$$|x^3 - a^3| < 3(1 + |a|)^2\delta.$$

Solution: We start with a bit of algebra:

$$\begin{aligned} |x^3 - a^3| &= |(x - a)(x^2 + ax + a^2)| \\ &= |x - a||x^2 + ax + a^2| \\ &\le |x^2 + ax + a^2|\delta && \text{(Using that } |x - a| < \delta) \\ &\le \left(|x^2| + |ax| + |a^2|\right)\delta && \text{(Triangle inequality)} \\ &= \left(|x|^2 + |a||x| + |a|^2\right)\delta && \text{(Using } |AB| = |A||B|) \end{aligned} \tag{4}$$

We now use use that "add and subtract" trick, which will come up at least 173 times during the term.

$$\begin{aligned} |x| &= |x - a + a| \\ &\le |x - a| + |a| && \text{(Triangle inequality)} \\ &< \delta + |a| && \text{(As } |x - a| < \delta) \\ &\le 1 + |a| && \text{(As } \delta \le 1) \end{aligned}$$

We also have the trivial inequality

$$|x| < |a| + 1.$$

Using these inequalities we have

$$\begin{aligned}|x|^2 + |a||x| + |a|^2 &\le (1+|a|)^2 + (1+|a|)(1+|a|) + (1+|a|)^2 \\ &= 3(1+|a|)^2.\end{aligned}$$

Finally using this in (4) we get

$$|x^3 - a^3| < 3(1+|a|)^2\delta$$

as required. □

Problem 2.29. Show that if $|y-c| < \delta \le 2$, then

$$|y^4 - c^4| \le 4(2+|c|)^3\delta$$

□

Problem 2.30. Assume $|a-x| < \delta$, $|y-b| < \delta$, and $\delta \le 1$. Show that

$$|xy - ab| < (1+|a|+|b|)\delta$$

Hint: Here is a start

$$\begin{aligned}|xy-ab| &= |xy - ay + ay - ab| && \text{(Adding and subtracting trick)} \\ &= |(x-a)y + a(y-b)| \\ &\le |x-a||y| + |a||y-b| && \text{(Triangle inequality)} \\ &\le \delta|y| + |a|\delta && (|x-a| < \delta \text{ and } |y-b| < \delta) \\ &= (|y| + |a|)\delta\end{aligned}$$

and the rest is left to you. □

Example 2.32. If $c \ne 0$ and $|z-c| < \delta < |c|/2$, show $z \ne 0$ and that

$$\left|\frac{1}{z} - \frac{1}{c}\right| < \frac{2\delta}{|c|^2}$$

Solution: First

$$\begin{aligned}|z| &= |c + (z-c)| && \text{(Add and subtract trick)} \\ &\ge |c| - |z-c| && \text{(reverse triangle inequality)}\end{aligned}$$

We are given that

$$|z-c| < \frac{|c|}{2}.$$

Thus

$$-|z-c| \ge -\frac{|c|}{2}.$$

Using this with what we have already done gives

$$|z| \ge |c| - |z-c| \ge |c| - \frac{|c|}{2} = \frac{|c|}{2}.$$

This implies $z \ne 0$. For future use note this also implies

$$\frac{1}{|z|} \le \frac{2}{|c|}.$$

Now

$$\begin{aligned} \left|\frac{1}{z} - \frac{1}{c}\right| &= \left|\frac{c-z}{zc}\right| \\ &= \frac{1}{|c|}\frac{1}{|z|}|c-z| \\ &< \frac{1}{|c|}\frac{1}{|z|}\delta && \text{(As } |z-c|<\delta\text{)} \\ &\le \frac{1}{|c|}\frac{2}{|c|}\delta && \text{(As } 1/|z| < 2/|c|\text{)} \\ &= \frac{2\delta}{|c|^2} \end{aligned}$$

as required. □

Problem 2.31. Assume $|x-a| < \delta \le |a|/2$. Show

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2}$$

and

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \frac{10\delta}{|a|^3}$$ □

Let $a, b \in \mathbb{F}$. Then define

$$\max(a,b) = \begin{cases} a, & \text{if } a \ge b; \\ b, & \text{if } b > a. \end{cases}$$

$$\min(a,b) = \begin{cases} a, & \text{if } a \le b; \\ b, & \text{if } b < a. \end{cases}$$

Proposition 2.33. *For $a, b \in \mathbb{F}$*

$$\max(a,b) = \frac{a+b+|a-b|}{2}$$

$$\min(a,b) = \frac{a+b-|a-b|}{2}$$

Problem 2.32. Prove this. □

Example 2.34. Solve the inequality $x^2 + 6x + 5 \le 0$.

Solution: Like many things involving quadratic polynomials completing the square is a smart thing to do. Adding 4 to both sides of the inequality gives

$$x^2 + 3x + 9 \le 4.$$

This can be rewritten as

$$(x+3)^3 \le 2^2$$

which is equivalent to

$$|x+3| \le 2.$$

This in turn is equivalent to

$$-3-2 \leq x \leq -3+2$$

and therefore the solution set is the interval

$$[-5,-1] = \{x : -5 \leq x \leq -1\}.$$

□

Problem 2.33. Solve the following inequalities

(a) $5x - 9 < 7x + 21$.
(b) $x^2 - 10x + 9 < 16$.
(c) $\dfrac{x+2}{x-2} \leq 5$. □

2.3. **The least upper bound axiom.** Let $\mathbb{F}$ be an ordered field and let $S \subseteq \mathbb{F}$ and assume $S \neq \varnothing$. Then S is ***bounded above*** if and only if there is a $b \in S$ such that

$$s \leq b \quad \text{for all} \quad s \in S.$$

Any such b is an ***upper bound*** for S. Likewise S is ***bounded below*** if there is an $a \in \mathbb{F}$ with

$$s \geq a \quad \text{for all} \quad s \in S,$$

and a is a ***lower bound*** for S.

Note that not every subset of $\mathbb{F}$ will have an upper bound or a lower bound. For example if $S = \mathbb{F}$, then S has no upper or lower bound. And in the rational numbers the integers, $\mathbb{Z}$, has no upper or lower bounds.

Definition 2.35. If $\varnothing \neq S \subseteq \mathbb{F}$, then c is a ***least upper bound*** (or ***supremum***) of S if c is an upper bound for S and $c \leq b$ for all upper bounds, b, of S. □

Proposition 2.36. *If S has a supremum, then it is unique.*

Proof. Let c and c' be supremums (i.e. least upper bounds) for S. Then by definition both the inequalities $c \leq c'$ and $c' \leq c$ hold. This implies $c = c'$. □

If S has a supremum, then we denote it by $\sup(S)$. In older books and articles, such as our text, this is often written as $\operatorname{lub}(S)$ or $\text{l. u. b.}(S)$.

Problem 2.34. Let $S \subseteq \mathbb{F}$, with $S \neq \varnothing$.

(a) Define what it means for a to be a ***lower bound*** for S.
(b) Define what it means for c to be a ***greatest lower bound*** (or ***infinmum***) of S.
(c) Prove that if S has an infinmum, that it is unique. It will be denoted by $\inf(S)$ (or $\operatorname{glb}(S)$, or $\text{g. l. b.}(S)$). □

If $\varnothing \neq S \subseteq \mathbb{F}$, then b is a ***largest***, or ***maximum***, of S if and only if $b \in S$ and $s \leq b$ for all $s \in S$. A ***smallest*** or ***minimum*** of S is defined similarly. If S has a maximum, then it is denoted by $\max(S)$. Likewise if it has a minimum, it is denoted by $\min(S)$.

Problem 2.35. Show that if S has a maximum, then it it has a supremum and

$$\sup(S) = \max(S).$$

(And there is a similar result for sets with a minimum.) □

Proposition 2.37. *Let $S_1, S_2, , \ldots, S_n \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. If each S_j has a maximum, then so does the union $S = S_1 \cup S_2 \cup \cdots \cup S_n$ and*

$$\max(S) = \max(\max(S_1), \max(S_2), \ldots, \max(S_n)).$$

Proof. Let $m_j = \max(S_j)$ and set $m = \max(m_1, m_2, \ldots, m_n)$. Then we want to show $\max(S) = m$.

We first show $m \in S$. By the definition of m as the maximum of the m_j's we have $m = m_j$ for some j. Then $m = m_j \in S_j \subseteq S$ and so $m \in S$.

Let $s \in S$, then $s \in S_i$ for some i by the definition of the union. As $m_i = \max(S_i)$ we have $s \le m_i$ and by the definition of $m = \max(m_1, m_2, \ldots, m_n)$ we have $s \le m_j \le m$. Therefore $m = \max(S)$. □

Example 2.38. In the real numbers and for $j = 1, 2, 3, \ldots$ let $S_j = (-\infty, -1/j]$. This set has the maximum: $\max(S_j) = -1/j$. But (as we will show shorty) the union is

$$S := \bigcup_{j=1}^{\infty} S_j = (-\infty, 0)$$

and this set has no maximum. Therefore it is important in Proposition 2.37 that the union is finite. Anther easy example where the result breaks down for an infinite union is letting $S_j = \{j\}$ just be the set just containing the one element j. Then

$$S := \bigcup_{j=1}^{\infty} S_j = \{1, 2, 3, \ldots\}$$

and this set is not even bounded above and so does not even have a supremum. □

Proposition 2.39. *If $\varnothing \ne S \subseteq \mathbb{F}$ is finite, then S has both a maximum and a minimum.*

Proof. Let $S = \{s_1, s_2, \ldots, s_n\}$ and $S_j = \{s_j\}$. Then each S_j has $\max(S_j) = s_j$ and $S = S_2 \cup S_2 \cup \cdots S_n$. So the this is a corollary to Proposition 2.37 □

Theorem 2.40 (Least upper bound property)**.** *In the real numbers, $\mathbb{R}$, every nonempty set that is bounded above has a least upper bound.* □

Unfortunately we will not prove this result in this class. If is not much harder than results that we will prove, but is long and drawn out. The longest part of this is giving a precise construction of the real numbers. This can be done in several ways. The Wikipedia article

`https://en.wikipedia.org/wiki/Construction_of_the_real_numbers` has a good discussion of several of these constructions. This article closes with the quote, which applies to most of the presentations of these constructions, "The details are all included, but as usual they are tedious and not too instructive". If you really want to see all the details of at least one construction I have written up notes *Construction of the real numbers* of Cantor's method of construing the reals from the rationals. This comes to 15 pages and definitely confirms the quote above about being tedious and not too constrictive.

One ideation that the least upper bound property is central is the following theorem, which we will also not prove in these notes (but is proven in the notes mentioned above).

Theorem 2.41 (Uniqueness of the real numbers)**.** *Let $\mathbb{F}$ be an ordered field where every set that is bounded above has a least upper bound. Then $\mathbb{F}$ is isomorphic to the real numbers.* □

Thus the least upper bound property is in some sense the defining property of the real numbers.

There is nothing special about upper bounds:

Theorem 2.42 (Greatest lower bound property)**.** *In the real numbers every nonempty set that is bounded below has a greatest lower bound.*

Problem 2.36. Prove this. *Hint:* Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Let b be a lower bound. Let $-S$ be

$$-S := \{-s : s \in S\}.$$

Show $-b$ is an upper bound for $-S$ and therefore S is bounded above. Therefore by the least upper bound property of the real numbers $-S$ has a least upper bound $c = \sup(-S)$. Show that $-c$ is a greatest lower bound for S. □

2.3.1. *Some applications of the least upper bound principle.* We start by showing that the natural numbers have no upper bound, a statement that goes back to Archimedes.

Proposition 2.43 (Archimedes' axiom (big version))**.** *For any real number x there is a natural n with $x < n$.*

Problem 2.37. Prove this. *Hint:* Toward a contradiction assume that there is a real number x such that for all $n \in \mathbb{N}$ we have $n \leq x$. This means that the set $\mathbb{N}$ is bounded above and therefore by the least upper bound property that $\mathbb{N}$ has a least upper bound $b = \sup(\mathbb{N})$. Thus for all natural numbers n

$$n \leq b.$$

But for $n \in \mathbb{N}$ the number $n + 1$ is also a natural number and thus for all $n \in \mathbb{N}$

$$n + 1 \leq b,$$

and therefore for all $n \in \mathbb{N}$ we have

$$n \leq b - 1.$$

Use this to derive a contradiction. $\square$

Proposition 2.44. *Let $a > 1$ be a real number. Show that for any real number x, there is a natural number n such that $a^n > x$.*

Problem 2.38. Prove this. *Hint:* If this is false, then the set $S = \{a^n : n \in \mathbb{N}\}$ has an upper bound. Derive a contradiction along the lines of the proof of Proposition 2.43. $\square$

Proposition 2.45 (Archimedes' axiom (small version))**.** *Let $a > 0$ be a positive real number. There there is a natural number, n, such that $1/n < a$.*

Proof. By the first version of Archimedes's axiom there is a natural number n with $n > 1/a$. But then $1/n < a$. $\square$

Problem 2.39. Give anther proof of Proposition 2.45 as follows. Let $S = \{1/n : n \in \mathbb{N}\}$. This is bounded below by 0. We want to show $\inf(S) = 0$. Toward a contradiction assume $c := \inf(S) > 0$. Then for all $n \in \mathbb{N}$ we have $c \leq 1/n$. But if $n \in \mathbb{N}$, then so is $2n$ and therefore $c \leq 1/(2n)$ which implies $2c \leq 1/n$, that is $2c$ is also a lower bound for S. Use this to contradict that c is the greatest lower bound. $\square$

Proposition 2.46. *Let a be a real number with $0 < a < 1$. Then for any positive real number x, there is a natural number n such that $a^n < x$.*

Problem 2.40. Prove this as a corollary to Proposition 2.44. $\square$

Remark 2.47. Let $x \in \mathbb{R}$. Then we want to define $\lfloor x \rfloor$, the greatest integer in x. That is the integer n with $n \leq x < n+1$. Here is the naive, proof, which is unfortunately not mathematically rigorous. Let $S = \{k \in \mathbb{Z} : k \leq x\}$. Let n be the maximum of S. Then $n \in S$ implies $n \leq x$. Since n is the largest element of S, $n + 1 \notin S$, which means that $x < n + 1$. So we have $n \leq x < n+1$. The problem is that S is infinite and so there is no guarantee that S has a largest element. In fact there are ordered fields where the set, $\mathbb{Z}$, of integers is bounded above, but $\mathbb{Z}$ does not have a least upper bound. This type of problem is why the proof of the nest result is a bit messy.

Proposition 2.48. *Let $S \subseteq \mathbb{Z}$ be a set of integers that is bounded below. Then S has a smallest element. That is $\min(S)$ exists.*

Problem 2.41. Prove this. *Hint:* As S is bounded below, it has a greatest lower bound, $\inf(S)$. Let $a = \inf(S)$. We are done if we can show $a \in S$. Assume, toward a contradiction, that $a \notin S$. Then show that there is an element $s_1 \in S$ with $a - 1/2 < s_2 < a$ (otherwise $a - 1/2$ would be a upper bound for S less that the greatest upper bound. This step does use that $a \notin S$ explain why). But the same reasoning there is $s_2 \in S$ with $s_1 < s_2 < a$. But then $s_1, s_2 \in (a - 1/2, a)$ and thus $|s_1 - s_2| < 1/2$. But $s_1 \neq s_2$ and thus $|s_1 - s_2| \geq 1$ as $|s_1 - s_2|$ is a positive integer. This gives the required contradiction.

Proposition 2.49 (Existence of greatest integer)**.** *For any real number x there is a unique integer n such that*

$$n \le x < n+1.$$

Problem 2.42. Prove this. *Hint:* Let $S := \{k \in \mathbb{Z} : x < k\}$ be the set of integers that are greater than x. Show that Archimedes Axiom implies $S \neq \varnothing$. By Proposition 2.48 S has a smallest element, $m = \inf(S) \in S$. Show that $n := m - 1$ satisfies $n \le x < n+1$.

To prove uniqueness, assume that n_1, n_2 are integers with ¡F3¿$n_2 \le x < n_2 + 1$ and $n_2 \le x < n_2 + 1$ and show this implies $-1 < n_2 - n_1 < 1$ and as n_1 and n_2 are integers this give $n_1 = n_2$. □

From now on we use the notation

$$\lfloor x \rfloor = \text{The unique integer } n \text{ with } n \le x < n+1.$$

The integer $\lfloor x \rfloor$ is called the ***greatest integer*** in x or the ***floor*** of x. Let

Proposition 2.50. *Between any two real numbers there is a rational number. (That is if $a < b$ are real numbers, there is a rational number, r, with $a < r < b$.)*

Problem 2.43. Prove this. *Hint:* By one form of Archimedes axiom there is a natural number N with

$$\frac{1}{N} < (b-a).$$

Let $n = \lfloor Na \rfloor$. Then

$$n \le Na < n+1.$$

Show

$$a < \frac{n+1}{N} < b$$

and therefore the rational number

$$r = \frac{n+1}{N}$$

does the trick. □

Proposition 2.51. *Between any two rational numbers there is an irrational number.*

Problem 2.44. Prove this. □

The following may look a little silly at first, but variants on it will be used repeatedly during the term to show that two numbers are equal. The first example of this is the proof of Theorem 2.55 below.

Proposition 2.52. *Let $y_0, y_1 \in \mathbb{R}$ and assume that there is a number $M > 0$ such that for all $\varepsilon > 0$*

$$|y_1 - y_0| \le M\varepsilon. \tag{5}$$

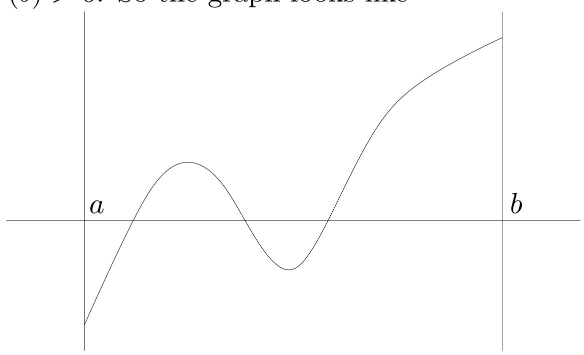
Then $y_0 = y_1$.

Problem 2.45. Prove this. *Hint:* Towards a contradiction assume that $y_0 \neq y_1$. Then let

$$\varepsilon = \frac{|y_1 - y_0|}{2M}$$

in the inequality (5) and show this leads to a contradiction. □

We now show that the least upper bound principle lets us show that reasonable equations have solutions. Consider a function $f : [a, b] \to \mathbb{R}$ with $f(a) < 0$ and $f(b) > 0$. So the graph looks like

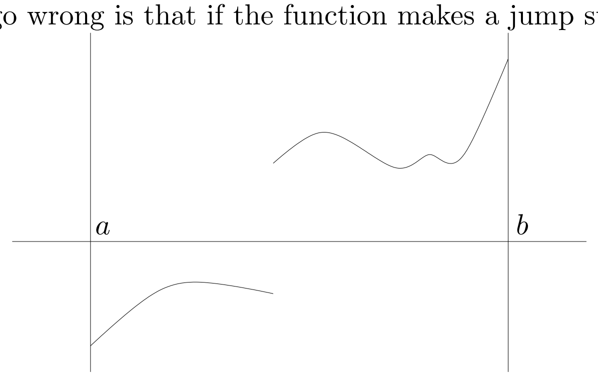


where the graph starts below the x-axis (at $x = a$) and ends up above the axis (at $x = b$). Then if there is any justice the graph will cross the axis somewhere between a and b and thus there is at least one point $\xi \in (a, b)$ with $f(\xi) = 0$. That is the equation

$$f(x) = 0$$

has a solution in the interval (a, b).

What can go wrong is that if the function makes a jump such as



So we need a condition that rules out jumps. Later in the term we will define what it means for a function to be continuous and show that for continuous functions that there are no jumps. Here we will use a less general condition.

Definition 2.53. Let $S \subseteq \mathbb{R}$. Then a function $f : S \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M \geq 0$ such that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

for all $x_1, x_2 \in S$. The constant M is a ***Lipschitz constant*** for f.

Note the Lipschitz is not unique for if M is a Lipschitz constant the so is M_1 for any $M_1 > M$.

To understand what the Lipschitz condition means first consider the graph the subset of the plane defined by the curve $|y-y_0| = M|x-x_0|$ as in Figure 1

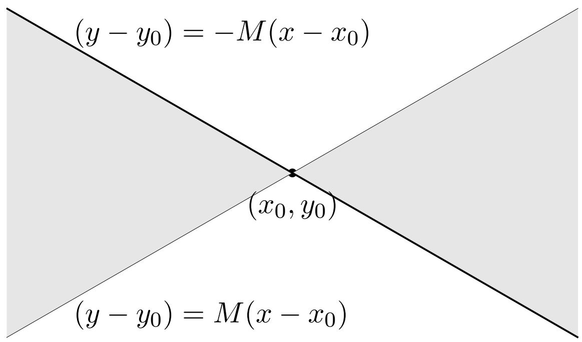


FIGURE 1. The two lines are the lines through (x_0, y_0) with slope $\pm M$. The shaded "butterfly" is the set of points (x, y) with $|y - y_0| \leq M|x - x_0|$. Call this the ***butterfly of slope M at*** (x_0, y_0) (this is not standard terminology and will only be for the next page or two).

A function is Lipschitz if and only if for each point $(x_0, y_0) = (x_0, f(x_0))$ the graph of f stays inside the butterfly centered at (x_0, y_0). See Figure 2.

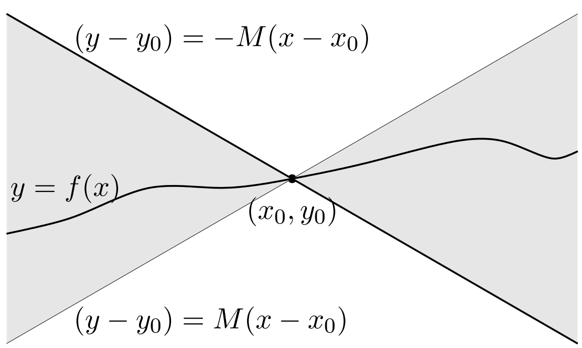


FIGURE 2. The function f is Lipschitz because for any x_0 in the domain of f, if $y_0 = f(x_0)$ then the graph of f is inside the butterfly of slope M at (x_0, y_0).

Proposition 2.54. *If $f\colon [a, b] \to \mathbb{R}$ is Lipschitz with Lipschitz constant M then for any $x, x_0 \in [a, b]$ the inequalities*

$$-M|x - x_0| \leq f(x) - f(x_0) \leq M|x - x_0|$$

and

$$f(x_0) - M|x - x_0| \le f(x) \le f(x_0) + M|x - x_0|$$

hold.

Problem 2.46. Prove this. *Hint:* Proposition 2.24 will save some work. □

Theorem 2.55 (Lipschitz intermediate value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a function such that*

$$f(a) \le 0 \quad \textit{and} \quad f(b) \ge 0$$

and such that there is a $M > 0$ such that for all $x_1, x_2 \in [a,b]$ the inequality

$$|f(x_2) - f(x_1)| \le M|x_2 - x_1|$$

*holds. (A function that satisfies this condition for some M is called a **Lipschitz function**.) Then there is a number $\xi \in [a,b]$ with*

$$f(\xi) = 0.$$

Problem 2.47. Prove this. *Hint:* First we use the least upper bound principle to get a candidate for ξ. Let

$$S = \{x \in [a,b] : f(x) \le 0\}.$$

Then $S \ne \varnothing$ (as $a \in S$) and S is bounded above (by b) and therefore

$$\xi = \sup(S)$$

exists. Let $\varepsilon > 0$. (Very many of our proofs during the term will start with the phrase "Let $\varepsilon > 0$".)

(a) If $f(a) = 0$ or $f(b) = 0$, we are done, so assume $f(a) < 0$ and $f(b) > 0$.

(b) Explain why there is a x_1 with

$$\xi - \varepsilon < x_1 \le \xi \quad \text{and} \quad f(x_1) \le 0.$$

(*Hint:* $[\xi - \varepsilon] \cap S \ne \varnothing$, for if not $\xi - \varepsilon$ would be an upper bound for S.)

(c) Note

$$f(\xi) = f(x_1) + (f(\xi) - f(x_1)) \le 0 + |f(\xi) - f(x_1)|$$

and use this to show

$$f(\xi) \le M\varepsilon.$$

(d) Explain why there is a $x_2 \in [\xi, \xi + \varepsilon]$ with

$$f(x_2) > 0.$$

(*Hint:* If $x_2 > \xi$, then $\xi \notin S$ and thus $f(x_2) > 0$.)

(e) Use that $f(x_2) > 0$ and $|\xi - x_2| < \varepsilon$ to show

$$f(\xi) \ge -M\varepsilon.$$

(f) Combine steps (d) and (f) to get that

$$|f(\xi)| = |f(\xi) - 0| \le M\varepsilon.$$

(g) This works for all $\varepsilon > 0$. Now use Proposition 2.52 to finish the proof. $\square$

We now use this to show that positive real numbers have square roots.

Proposition 2.56. *Let $c > 0$ be a positive real number. Then c has a positive square root.*

Proof. Let f be the function

$$f(x) = x^2 - c$$

and let

$$a = 0 \quad \text{and} \quad b = c + 1.$$

Then

$$f(a) = f(0) = -c < 0$$

and

$$f(b) = (c+1)^2 - c = c^2 + c + 1 > 0.$$

Also if $x_1, x_2 \in [a, b]$ we have

$$\begin{aligned} |f(x_2) - f(x_2)| &= |x_2^2 - x_1^2| \\ &= |x_2 + x_1||x_2 - x_1| \\ &\leq (|x_2| + |x_1|)|x_2 - x_1| && \text{(triangle inequality)} \\ &\leq 2b|x_2 - x_1| && \text{(as } |x_1|, |x_2| \leq b\text{)} \\ &= M|x_2 - x_1| \end{aligned}$$

where $M = 2b$. Therefore $f(x)$ satisfies the hypothesis of our version of the intermediate value theorem and so there is a $\xi \in (a, b)$ with $f(\xi) = 0$. But then $\xi^2 = c$ and therefore c has a square root. $\square$

You should read the proof that positive numbers have a square root in pages 28–29 of the text, to see a different version of this proof.

Proposition 2.57. *Let $c > 0$. Then c has a cube root.*

Problem 2.48. Prove this along the lines of the proof of Proposition 2.56. $\square$

2.4. **Some more inequalities with application to vector algebra.** Maybe the most basic inequality for real numbers is that squares of numbers are nonnegative:

$$x^2 \geq 0 \quad \text{for all } x \in \mathbb{R} \text{ with equality if and only if } x = 0.$$

One of the best known inequalities that is a direct consequence of this is

Proposition 2.58 (Arthemetic geometric mean inequality)**.** *For any two positive real numbers a and b the inequality*

$$\sqrt{ab} \leq \frac{a+b}{2}$$

holds with equality holding if and only if $a = b$.

Problem 2.49. Prove this. *Hint:* Start by noting

$$\frac{a+b}{2} - \sqrt{ab} = \frac{\big(\sqrt{a} - \sqrt{b}\big)^2}{2} \geq 0. \qquad \square$$

Problem 2.50. Show for any real number $a > 0$ that

$$a + \frac{1}{a} \geq 2$$

with equality if and only if $a = 1$. *Hint:* $(\sqrt{a} - 1/\sqrt{a})^2 \geq 0$. $\square$

We now review a little bit of vector algebra and apply use that the sum of squares are positive to derive some useful inequalities. Let $\mathbb{R}^n$ be the set of n-tuples of real numbers. That is

$$\mathbb{R}^n := \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in \mathbb{R}\}.$$

Elements of $\mathbb{R}^n$ are called ***vectors***. And in this context real numbers are often called ***scalars***.

I assume that you know how to add and subtract vectors and to multiple them by scalars. If

$$\mathbf{a} = (a_1, a_2, \ldots, a_n), \ \mathbf{b} = (b_1, b_2, \ldots, b_n)$$

are vectors in $\mathbb{R}^n$ then their ***inner product*** is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{j=1}^{n} a_j b_j.$$

Note

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + \cdots + a_n^2$$

which is a sum of squares of real numbers. Thus

$$\mathbf{a} \cdot \mathbf{a} \geq 0 \quad \text{with equality if and only if } \mathbf{a} = \mathbf{0}.$$

As $\mathbf{a} \cdot \mathbf{a} \geq 0$, it has a unique positive square root. Define

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

Or what is the same thing $\|\mathbf{a}\|$ is the positive real number with

$$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}.$$

The number $\|\mathbf{a}\|$ is called the ***length*** or ***norm*** of $\mathbf{a}$,

The following records some basic properties of the inner product and and norms.

Proposition 2.59. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *Then*

(a) The distributive law

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}$$

holds.

(b) If c is a scalar, then

$$(c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b}) = c(\mathbf{a} \cdot \mathbf{b}).$$

(c) If s is a scalar, then

$$\|s\mathbf{a}\| = |s|\|\mathbf{a}\|$$

so that if $s \geq 0$, then $\|s\mathbf{a}\| = s\|\mathbf{a}\|$.

(d) If s and t are scalars, then we can expand $\|s\mathbf{a} + t\mathbf{b}\|^2$ in the natural way. That is

$$\|s\mathbf{a} + t\mathbf{b}\|^2 = s^2\|\mathbf{a}\|^2 + 2st\mathbf{a}\cdot\mathbf{b} + t^2\|\mathbf{b}\|^2.$$

Problem 2.51. Prove this. □

Lemma 2.60. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then*

$$0 \leq \Big\| \|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b} \Big\|^2 = 2\|\mathbf{a}\|\|\mathbf{b}\|\Big(\|\mathbf{a}\|\|\mathbf{b}\| + \mathbf{a}\cdot\mathbf{b}\Big)$$
$$0 \leq \Big\| \|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b} \Big\|^2 = 2\|\mathbf{a}\|\|\mathbf{b}\|\Big(\|\mathbf{a}\|\|\mathbf{b}\| - \mathbf{a}\cdot\mathbf{b}\Big)$$

Problem 2.52. Prove this. *Hint:* Use $s = \|\mathbf{a}\|$ and $t = \pm\|\mathbf{b}\|$ in part (d) of Proposition 2.59. □

Theorem 2.61 (The Cauchy-Schwartz inequality)**.** *If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then*

$$\mathbf{a}\cdot\mathbf{b} \leq \|\mathbf{a}\|\|\mathbf{b}\|$$

and

$$|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|.$$

Problem 2.53. Use Lemma 2.60 to prove this. □

In $\mathbb{R}^2$ and $\mathbb{R}^3$ there is a somewhat more geometric way to think of the Cauchy-Schwartz inequality. In your vector calculus class you showed that

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. (This is equivalent to the law of cosines from trigonometry.) As $\cos(\theta) \leq 1$ this gives

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta) \leq \|\mathbf{a}\|\|\mathbf{b}\|.$$

While this is more intuitive than our proof here, it is using concepts that we have yet to define (such as angles and the cosine function). But thinking of $\mathbf{a}\cdot\mathbf{b}$ as $\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$ is an easy way to remember what the Cauchy-Schwartz says.

Theorem 2.62 (The triangle inequality for vectors)**.** *For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ the inequality*

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$

holds.

Proof. We have

$$\begin{aligned} \|\mathbf{a}+\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2 && \text{(Prop. 2.59 part (d))} \\ &\le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 && \text{(Cauchy-Schwartz)} \\ &= \big(\|\mathbf{a}\| + \|\mathbf{b}\|\big)^2 \end{aligned}$$

Taking square roots now gives the required inequality. □

Proposition 2.63. *For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the **reverse triangle inequality***

$$\Big| \|\mathbf{a}\| - \|\mathbf{b}\| \Big| \le \|\mathbf{a} - \mathbf{b}\|$$

holds.

Problem 2.54. Prove this. □

3. Metric Spaces.

Definition 3.1. A ***metric space*** is a nonempty set E with a function $d\colon E \times E \to [0, \infty)$ such that for all $p, q, r \in E$ the following hold

(a) $d(p, q) \ge 0$,
(b) $d(p, q) = 0$ if and only if $p = q$,
(c) $d(p, q) = d(q, p)$, and
(d) $d(p, r) \le d(p, q) + d(q, r)$. □

The function d is called the ***distance function*** on E. The condition $d(p, q) = d(q, p)$ is that the distance between points is ***symmetric***. The inequality $d(p, r) \le d(p, q) + d(q, r)$ is the ***triangle inequality***.

The most basic example if a metric space is when $E \subseteq \mathbb{R}$ is a nonempty subset of the real numbers, $\mathbb{R}$, and the distance is defined by

$$d(p, q) = |p - q|.$$

Problem 3.1. Show that this makes E into a metric space. □

We have seen that if $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are points in $\mathbb{R}^n$ and we define the ***length*** or ***norm*** of p to be

$$\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$$

then the inequality

$$\|p + q\| \le \|p\| + \|q\|$$

holds.

Proposition 3.2. *Let E be a nonempty subset of $\mathbb{R}^n$ and for $p, q \in E$ let*

$$d(p, q) = \|p - q\|.$$

Then E with the distance function d is a metric space.

Problem 3.2. Prove this. □

Here are some inequalities that we will be using later.

Proposition 3.3 (Reverse triangle inequality)**.** *Let E be a metric space with distance function d and let $x, y, z \in E$. Then*

$$|d(x,y) - d(x,z)| \le d(y,z).$$

Problem 3.3. Prove this. Then draw a picture in the case of $\mathbb{R}^2$ with its standard distance function showing why this inequality is reasonable. □

Proposition 3.4. *Let E be a metric space with distance function d and $x_1, \dots, x_n \in E$. Then*

$$d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Problem 3.4. Prove this. Then draw a picture in the plane showing why this is reasonable. *Hint:* Induction. □

Definition 3.5. Let E be a metric space with distance function d. Let $a \in E$, and $r > 0$.

(a) The ***open ball*** of radius r centered at x is

$$B(a,r) := \{x : d(a,x) < r\}.$$

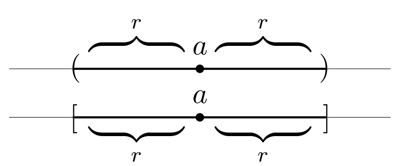
(b) The ***closed ball*** or radius r centered at a is

$$\overline{B}(a,r) := \{x : d(a,x) \le r\}.$$

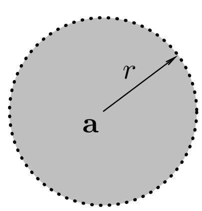
□

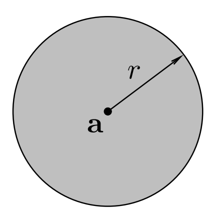
In the real numbers with their usual metric $d(x,y) = |x-y|$ the open and closed balls about a are intervals with center a:

$$B(a,r) = (a-r, a+r) =$$
$$\overline{B}(a,r) = [a-r, a+r] =$$



In the plane, $\mathbb{R}^2$, with its usual metric $d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\|$, the open and closed balls about $\mathbf{a}$ are disks centered at $\mathbf{a}$.

$B(\mathbf{a}, r) =$ 

$\overline{B}(\mathbf{a}, r) =$ 

Definition 3.6. Let E be a metric space with distance function d. Then $S \subseteq E$ is an ***open set*** if and only if for all $x \in S$ there is an $r > 0$ such that $B(x,r) \subseteq S$. □

This can be restated by saying that S is open if and only if each of its points is the center of an open ball contained in S. See Figure 3.

Proposition 3.7. *In any metric space E, the sets E and $\varnothing$ are open.* □

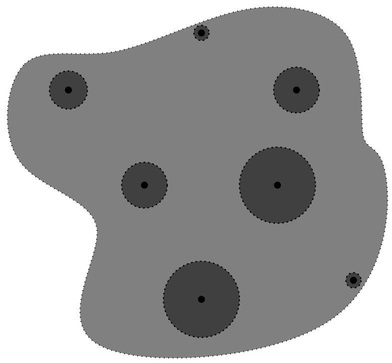


FIGURE 3. A set is open if and only if each of its points is the center of an open ball contained in the set.

Proof. Let $p \in E$, then for any $r > 0$ we have $B(p,r) = \{x \in E : d(x,p) < r\} \subseteq E$. Thus E contains not only some open ball about p, it contains every open ball about p. Therefore E is open.

That $\varnothing$ is open is a case of a vicious implication. To see this consider the statement

$$p \in \varnothing \text{ and } r > 0 \implies B(p,r) \subseteq \varnothing.$$

If this statement is true, then $\varnothing$ satisfies the definition of beiing open. But this is a true statement as an implication $P \implies Q$ is true whenever the hypotheses, P, is false. And the hypothesis "$p \in \varnothing$ and $r > 0$" is false as "$p \in \varnothing$" is false. □

Proposition 3.8. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the open ball $B(x,r)$ is an open set.*

Problem 3.5. Prove this. *Hint:* Let $x \in B(a,r)$. Then $d(a,x) < r$. Set $\rho := r - d(a,x) > 0$ and show $B(x,\rho) \subseteq B(a,r)$ □

Proposition 3.9. *In the real numbers, $\mathbb{R}$, with their standard metric, the open intervals (a,b) are open.*

Problem 3.6. Prove this. □

Proposition 3.10. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the compliment, $\mathcal{C}(\overline{B}(a,r))$, of the closed ball $\overline{B}(a,r)$ is open.*

Problem 3.7. Prove this. *Hint:* If $x \in \mathcal{C}(B(a,r))$, then $d(x,a) > r$. Let $\rho := d(a,x) - r > 0$ and show $B(a,\rho) \subseteq \mathcal{C}(B(a,r))$. □

Proposition 3.11. *If U and V are open subsets of E, then so are $U \cup V$ and $U \cap V$.*

Proof. Let $x \in U \cup V$. Then $x \in U$ or $x \in V$. By symmetry we can assume that $x \in U$. Then, as U is open, there is an $r > 0$ such $B(x,r) \subseteq U$. But then $B(x,r) \subseteq U \subseteq U \cup V$. As x was any point of $U \cup V$ this shows that $U \cup V$ is open.

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. As $x \in U$ there is an $r_1 > 0$ such that $B(x, r_1) \subseteq U$. Likewise there is an $r_2 > 0$ such that $B(x, r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then

$$B(x, r) \subseteq B(x, r_1) \subseteq U \quad \text{and} \quad B(x, r) \subseteq B(x, r_2) \subseteq V$$

and therefore $B(x, r) \subseteq U \cap V$. As x was any point of $U \cap V$ this shows that $U \cap V$ is open. $\square$

Proposition 3.12. *Let E be a metric space.*

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Then the union $\bigcup_{i \in I} U_i$ is open.
(b) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of E. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Problem 3.8. Prove this. $\square$

Problem 3.9. In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open. $\square$

Definition 3.13. Let E be a metric space. Then a subset S of E is ***closed*** if and only if its compliment, $\mathcal{C}(S)$ is open. $\square$

Because the compliment of the compliment is the original set this implies that a set, S, is open if and only if its compliment $\mathcal{C}(S)$ is closed. Likewise a set, S, is closed if and only if its compliment $\mathcal{C}(S)$ is open.

Proposition 3.14. *In any metric space E the sets $\varnothing$ and E are both closed.*

Proof. We have seen the sets E and $\varnothing$ are open, thus their compliments $\mathcal{C}(E) = \varnothing$ and $\mathcal{C}(\varnothing) = E$ are closed. $\square$

Proposition 3.15. *If E is a metric space, $a \in E$, and $r > 0$, then the closed ball $\overline{B}(a, r)$ is closed.* $\square$

Proof. The compliment $\mathcal{C}(\overline{B}(a, r))$ is open by Proposition 3.10 $\square$

Problem 3.10. Show that in $\mathbb{R}$ with its usual metric the closed intervals are closed. $\square$

Proposition 3.16. *If E is a metric space, then every finite subset of E is closed.*

Problem 3.11. Prove this. *Hint:* Since a finite set is finite union of one point sets, it is enough to show that a one point set is close. So let $p \in E$ and $x \in \mathcal{C}(\{p\})$. Let $r = d(x, p)$ and show $B(x, r) \subseteq \mathcal{C}(\{p\})$. $\square$

Problem 3.12. In the real numbers show that the half open interval $[0, 1)$ is neither open or closed. $\square$

Problem 3.13. The integers, $\mathbb{Z}$, are a metric space with the metric $d(m, n) = |m - n|$. Note that for this metric space if $m \neq n$ that $d(m, n)$ is a nonzero positive integer and thus $d(m, n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n, r)$ is the one element set $B(n, r) = \{n\}$ and therefore $\{n\}$ is open.
(b) Every subset of $\mathbb{Z}$ is open. *Hint:* Let $S \subseteq \mathbb{Z}$, then $S = \bigcup_{n \in S} \{n\}$ and use Proposition 3.12 to conclude that S is open.
(c) Every subset of $\mathbb{Z}$ is closed. □

Proposition 3.17. *Let E be a metric space.*

(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Then the intersection $\bigcap_{i \in I} F_i$ is closed.
(b) Let $F_1, \ldots, F_n$ be a finite collection of closed subsets of E, then the union $U_1 \cup \cdots \cup U_n$ is closed.

Problem 3.14. Prove this. *Hint:* The correct way to do this is to deduce it directly from Proposition 3.12. For example to show that the union of two closed sets is closed, let F_1 and F_2 be closed. Then the compliments $\mathcal{C}(F_1)$ and $\mathcal{C}(F_1)$ are open and the intersection of two open sets is open. Therefore $\mathcal{C}(F_1) \cap \mathcal{C}(F_2)$ is open and thus the compliment of this set is closed. That is

$$F_1 \cup F_2 = \mathcal{C}(\mathcal{C}(F_1) \cap \mathcal{C}(F_2))$$

is closed. □

Definition 3.18. Let E be a metric space. Then a function $f : E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M \geq 0$ such that

$$|f(p) - f(q)| \leq M d(p, q) \quad \text{for all } p, q \in E.$$

for all $p, q \in E$. □

Proposition 3.19. *Let E be a metric space and $f : E \to \mathbb{R}$ a Lipschitz function. Then for all $c \in \mathbb{R}$ the sets*

$$f^{-1}\big[(c, \infty)\big] = \{p \in E : f(p) < c\}$$
$$f^{-1}\big[(-\infty, c)\big] = \{p \in E : f(p) > c\}$$

are open and the sets

$$f^{-1}\big[[c, \infty)\big] = \{p \in E : f(p) \geq c\}$$
$$f^{-1}\big[(-\infty, c]\big] = \{p \in E : f(p) \leq c\}$$

are closed.

Half of the proof. Assume that f satisfies $|f(p) - f(q)| \leq M d(p, q)$ for $p, q \in E$. We will show that $f^{-1}\big[(-\infty, c)\big]$ is open. We need to show that for any $q \in f^{-1}\big[(-\infty, c)\big]$ the set $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about q. As $q \in f^{-1}\big[(-\infty, c)\big]$ we have $f(q) < c$. Therefore

$$r = \frac{c - f(q)}{M}$$

is positive. Let $pe \in B(q,r)$. Then

$$\begin{aligned} f(p) &= f(q) + (f(p) - f(q)) \\ &\le f(q) + |f(p) - f(q)| && \text{(as } (f(p) - f(q)) \le |f(p) - f(q)|\text{)} \\ &\le f(q) + Md(p,q) && \text{(as } f \text{ is Lipschitz)} \\ &< f(q) + Mr && \text{(as } p \in B(q,r)\text{, so } d(p,q) < r\text{)} \\ &= f(q) + M\left(\frac{c - f(q)}{M}\right) && \text{(from our definition of } r\text{)} \\ &= c. \end{aligned}$$

Therefore if $p \in B(q,r)$ we have $f(p) < c$ and thus $B(q,r) \subseteq f^{-1}\big[(-\infty, c)\big]$. Whence $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about any of its points q and therefore $f^{-1}\big[(-\infty, c)\big]$ is open.

We now show $f^{-1}\big[[c, \infty\big] = \{p \in E : f(p) \ge c\}$ is closed. We know $f^{-1}\big[(-\infty, c)\big] = \{p \in E : f(p) < c\}$ is open. Its compliment is

$$\mathcal{C}\left(f^{-1}\big[(-\infty, c)\big]\right) = f^{-1}\big[[c, \infty)\big].$$

Therefore $f^{-1}\big[[c, \infty)\big]$ is the compliment of an open set, which means that $f^{-1}\big[[c, \infty)\big]$ is closed. □

Problem 3.15. Prove the other half of Proposition 3.19, that is show $f^{-1}\big[(c, \infty)\big]$ is open and $f^{-1}\big[(-\infty, c]\big]$ is closed. *Hint:* Rather than doing a bunch more computations with inequalities, note that if $g(p) = -f(p)$ is the negative of f, then show

$$\begin{aligned} f^{-1}\big[(c, \infty)\big] &= g^{-1}\big[(-\infty, -c)\big] \\ f^{-1}\big[(-\infty, c]\big] &= g^{-1}\big[[-c, \infty)\big] \end{aligned}$$

and that g is Lipschitz so that we have reduced it to the case we have already done. □

Proposition 3.20. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for any $c \in \mathbb{R}$ the set*

$$f^{-1}[c] = \{p \in E : f(p) = c\}$$

is a closed set.

Problem 3.16. Prove this. *Hint:* Write $f^{-1}[c]$ as the intersection of two closed sets. □

We can now give some more examples of open and closed sets in the plane, $\mathbb{R}^2$ where we are using the distance function $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$. Let $\mathbf{a} = (a_1, a_2)$ be a vector in $\mathbb{R}^2$ define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = a_1x_1 + a_2x_2 + b$$

where b is a real number. If $\mathbf{x} = (x_1, x_2)$ this can be written in vector form as

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

If $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ then

$$\begin{aligned} |f(\mathbf{p}) - f(\mathbf{q})| &= |\mathbf{a}\cdot\mathbf{p} + b - (\mathbf{a}\cdot\mathbf{q} + b)| \\ &= |\mathbf{a}\cdot(\mathbf{p}-\mathbf{q})| \\ &\le \|\mathbf{a}\|\|\mathbf{p}-\mathbf{q}\| && \text{(Cauchy-Schwartz)} \\ &= Md(\mathbf{p},\mathbf{q}) \end{aligned}$$

where $M = \|\mathbf{a}\|$. Thus f is a Lipschitz function.

Consider the case where $\mathbf{a} = (1, 0)$ and $b = 0$. Then for any $c \in \mathbb{R}$. In this case $f(x_1, x_2) = x_1$, or in slightly different notation $f(x, y) = x$. Therefore Proposition 3.19 implies the sets

$$\{(x, y) : x > c\}, \quad \{(x, y) : x < c\}$$

are open and that

$$\{(x, y) : x \ge c\}, \quad \{(x, y) : x \le c\}$$

are closed.

Problem 3.17. Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector and $c \in \mathbb{R}$.

(a) Show that the line

$$\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is closed.

(b) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by > c\}$$

is open (call such a half plane an ***open half plane***).

(c) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by \ge c\}$$

is closed (call such a half plane a ***closed half plane***).

(d) Show that the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x + y < 1\}$$

is an open set. *Hint:* Write this as the intersection of three open half planes.

(e) Show that the triangle

$$S = \{(x, y) : x, y \ge 0, x + y \le 1\}$$

is a closed subset of the plane. *Hint:* Write this as the interestion of three closed half planes. □

3.1. **Definition of limit in a metric space and some limits in $\mathbb{R}$.**

Definition 3.21. Let E be a metric space and $\langle p_n\rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots\rangle$ a sequence in E. Then

$$\lim_{n\to\infty} p_n = p$$

if and only if for all $\varepsilon > 0$ there is a $N > 0$ such that

$$n > N \implies d(p_n, p) < \varepsilon.$$

In the case we say that the sequence $\langle p_n\rangle_{n=1}^{\infty}$ ***converges*** to p. $\square$

Problem 3.18. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim_{n\to\infty} a_n = p$ also holds. $\square$

Here are some examples of working with limits in $\mathbb{R}$.

Example 3.22. If $\langle x_n\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} 5x_n = 5x$.

Proof. Let $\varepsilon > 0$. Note that

$$|5x_n - 5x| = 5|x_n - x|.$$

From the definition of $\lim_{n\to\infty} x_n = x$ there is a $N > 0$ such that

$$n > N \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{5}.$$

But then (multiply by 5)

$$n > N \quad \text{implies} \quad |5x_n - 5x| < \varepsilon.$$

But this is just the definition of $\lim_{n\to\infty} 5x_n = 5x$. $\square$

Proposition 3.23. *Let $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with*

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y.$$

Proof. Let $\varepsilon > 0$. Then from the definition of $\lim_{n\to\infty} x_n = x$, there is a $N_1 > 0$ such that

$$n > N_1 \quad \text{implies} \quad |x - x_n| < \frac{\varepsilon}{2}.$$

Likewise $\lim_{n\to\infty} y_n = y$ implies there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2}.$$

Set

$$N = \max\{N_1, N_2\}.$$

If $n > N$, then $n > N_1$ and $n > N_2$ and thus

$$|(x + y) - (x_n + y_n)| \le |x - x_n| + |y - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is

$$n > N \quad \text{implies} \quad |(x+y) - (x_n + y_n)| < \varepsilon$$

which is exactly the definition of $\lim_{n\to\infty}(x_n + y_n) = x + y$. $\square$

Proposition 3.24. *Let $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y.$$

Then for any real numbers a and b

$$\lim_{n\to\infty}(ax_n + by_n) = ax + by.$$

Problem 3.19. Prove this. $\square$

Proposition 3.25. *If $\langle x_n \rangle$ be a convergent sequence in $\mathbb{R}$. Then $\langle x_n \rangle$ is bounded. That is a constant M such that $|x_n| \leq M$ for all M.* $\square$

Problem 3.20. Prove this. *Hint:* Let $\varepsilon = 1$. Then there is a N such that

$$n > N \quad \text{implies} \quad |x - x_n| < 1.$$

Therefore is $n > N$ we have

$$|x_n| = |x + (x_n - x)| \leq |x| + |x - x_n| < |x| + 1.$$

This bounds all the terms with $n > N$. Let

$$M = \max\big\{|x| + 1, |x_1|, |x_2|, \ldots, |x_N|\big\}.$$

Then $|x_n| \leq M$ for all n, which shows that the sequence is bounded. $\square$

Theorem 3.26. *Let*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n y_n = xy.$$

Problem 3.21. Prove this. *Hint:* Start with

> **Scratch work that the no one else needs to see:** Our goal is to make $|x_n y_n - xy|$ small. We compute

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \qquad \text{(Adding and subtracting trick.)} \\ &\leq |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| \end{aligned}$$

> The factors $|x_n - x|$ and $|y_n - y|$ are both good in that we can make them small. The factor $|x|$ is independent of n and thus is not a problem. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent and thus bounded, so we bound the factor $|y_n|$. We now return to our regularly scheduled proof.

Let $\varepsilon > 0$. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent thus it is bounded. Therefore there is an M so that

$$|y_n| \leq M \quad \text{for all} \quad n.$$

As $\lim_{n\to\infty} x_n = x$ there There is a $N_1 > 0$ such that

$$n > N_1 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{2(M+1)}$$

and as $\lim_{n\to\infty} y_n = y$ there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2(|x|+1)}.$$

Now let $N = \max\{N_1, N_2\}$ and use the calculation from our scratch work to show

$$n > N \quad \text{implies} \quad |x_n y_n - xy| < \varepsilon$$

which completes the proof. □

Corollary 3.27. *If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$, then*

$$\lim_{n\to\infty} x_n^2 = x^2.$$

Proof. Use $\langle x_n \rangle = \langle y_n \rangle$ in Theorem 3.26. □

Proposition 3.28. *Let k be a positive integer and $\langle p_n \rangle$ a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} p_n = p$. Then*

$$\lim_{n\to\infty} p_n^k = p^k$$

Problem 3.22. Prove this. *Hint:* What is probably the easiest why is to use induction. □

Problem 3.23. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the quadratic polynomial $f(x) = ax^2+bx+c$ where a, b, c are constants. Let $\langle p_n \rangle$ be a convergent sequence, $\lim_{n\to\infty} p_n = p$. Then

$$\lim_{n\to\infty} f(p_n) = f(p).$$

□

Lemma 3.29. *Let $a \in \mathbb{R}$ with $a \neq 0$. Let $|x - a| < \dfrac{|a|}{2}$. Then*

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2},$$

$$\frac{1}{|x|} < \frac{2}{|a|},$$

and

$$\left| \frac{1}{x} - \frac{1}{a} \right| \leq \frac{2|x-a|}{|a|^2}.$$

Problem 3.24. Prove this. □

Proposition 3.30. *Let $\langle x_n \rangle$ be a sequence with* $\lim_{n\to\infty} = a$ *and* $a \neq 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}.$$

Problem 3.25. Prove this. *Hint:* First note there is a N_1 such that

$$n > N_1 \quad \text{implies} \quad |x_n - a| < \frac{|a|}{2}.$$

Now let $\varepsilon > 0$ There is also a N_2 such that

$$n > N_2 \quad \text{implies} \quad |x_n - a| < \frac{|a|^2}{2}\varepsilon.$$

Now let $N = \max\{N_1, N_2\}$ and use the last lemma to show that

$$n > N \quad \text{implies} \quad \left| \frac{1}{x_n} - \frac{1}{a} \right| < \varepsilon.$$

□

Proposition 3.31. *Let E be a metric space and $f\colon E \to \mathbb{R}$ be a Lipschitz map. (That is there is a constant M such that for all $p, q \in E$ the inequality $|f(p) - f(q)| \leq M d(p,q)$ holds.) Let $\langle p_n \rangle$ be a sequence in E with* $\lim_{n\to\infty} p_n = p$ *where $p \in E$. Then*

$$\lim_{n\to\infty} f(p_n) = f(p).$$

Problem 3.26. Prove this. □

3.1.1. *Limits and rational functions.* We show that limits play will with polynomials and rational functions. Recall a polynomial, $f(x)$, is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers. If $a_n \neq 0$, then the ***degree*** of $f(x)$ is $\deg f(x) = n$. The following is trivial, but useful in doing induction proofs involving polynomials.

Proposition 3.32. *Let $f(x)$ be a polynomial of degree $n \geq 1$. There there is a polynomial $g(x)$ of degree $(n-1)$ and a constant c such that*

$$f(x) = xg(x) + c.$$

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ Then factor out x from all the non-constant terms:

$$\begin{aligned} f(x) &= x\big(a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1\big) + a_0 \\ &= xg(x) + c \end{aligned}$$

where $g(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1$ and $c = a_0$. □

Theorem 3.33. *Let $\langle p_k \rangle_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$, say*

$$\lim_{k\to\infty} p_k = p.$$

Then for any polynomial $f(x)$

$$\lim_{k\to\infty} f(p_k) = f(p).$$

Problem 3.27. Prove this. *Hint:* Use induction on $\deg f(x)$. The base case is $\deg f(x) = 0$, that is $f(x) = a_0$ is a constant. This case $f(p_k) = a_0$ for all n and thus $\langle f(p_k)\rangle_{k=1}^{\infty}$ is a sequence of constants and so the result it true in this case. The case of $\deg f(x) = 1$ is also easy. In this case $f(x) = a_1x + a_0$. And from some some of our earlier results we have

$$\begin{aligned}\lim_{k\to\infty} f(p_k) &= \lim_{k\to\infty}(a_1p_k + a_0)\\ &= a_1p + a_0\\ &= f(p).\end{aligned}$$

Now do the induction step. Assume we know the result is true for polynomials of degree $(n-1)$ and let $f(x)$ be a polynomial of degree n. By Proposition 3.32 write

$$f(x) = xg(x) + c$$

where $g(x)$ is a polynomial of degree $(n-1)$ and c is a constant. Then by the induction hypothesis we have

$$\lim_{k\to\infty} g(p_k) = g(p).$$

Now use our earlier results about limits of products and sums to finish the induction step and complete the proof. $\square$

Lemma 3.34. *Let* $g(x)$ *be a polynomial and* $p \in \mathbb{R}$ *a point with* $g(p) \neq 0$. *Let* $\langle p_k\rangle_{k=1}^{\infty}$ *a sequence with*

$$\lim_{k\to\infty} p_k = p.$$

Then $g(p_k) \neq 0$ *for all but at most finitely many* k*'s, and this the sequence*

$$\left\langle \frac{1}{g(p_k)} \right\rangle_{k=1}^{\infty}$$

is defined for all but finitely many values of k *and*

$$\lim_{k\to\infty} \frac{1}{g(p_k)} = \frac{1}{g(p)}.$$

Problem 3.28. Prove this. *Hint:* First show that $g(p_k) \neq 0$ for all but finitely may k. One way to do this is to let $\varepsilon = |g(p)|/2$ in the definition of a limit. We know from Theorem 3.33 that

$$\lim_{k\to\infty} g(p_k) = g(p).$$

Let $\varepsilon = |g(p)|/2$ in the definition of $\lim_{k\to\infty} g(p_k) = g(p)$. to find a $N > 0$ such that

$$n > N \quad \text{implies} \quad |g(p_k) - g(p)| < |g(p)|/2.$$

Use this to show

$$n > N \quad \text{implies} \quad |g(k)| > |g(p)|/2$$

and therefore

$$n > N \quad \text{implies} \quad g(p_k) \neq 0.$$

You should now be able to use Proposition 3.30 to finish the proof.

A ***rational function*** is a function

$$h(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials, $g(x)$ is not identically zero and the domain of $h(x)$ is the set of points where $g(x) \neq 0$.

Theorem 3.35. *Let $\langle p_k \rangle_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$,*

$$\lim_{k\to\infty} p_k = p.$$

Let

$$h(x) = \frac{f(x)}{g(x)}$$

be a rational function with $g(p) \neq 0$. Then

$$\lim_{k\to\infty} h(p_k) = h(p).$$

Problem 3.29. Prove this by putting together Lemma 3.34 and Proposition 3.30. □

We can now do some limits you recall from calculus. For example let us compute

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 7}{4n^2 + 6}.$$

Divide the numerator and denominator of the fraction in the limit to get

$$\frac{3n^2 - 3n + 7}{4n^2 + 6} = \frac{3 - 3(1/n) + 7(1/n)^2}{4 + 6/(1/n)^2} = \frac{f(1/n)}{g(1/n)}$$

where $f(x)$ and $g(x)$ are the polynomials

$$f(x) = 3 - 3x + 7x^2 \quad g(x) = 4 + 6x^2.$$

And have seen that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore Theorem 3.35 gives

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 7}{4n^2 + 6} = \lim_{n\to\infty} \frac{f(1/n)}{g(1/n)} = \frac{f(0)}{g(0)} = \frac{3}{4}.$$

Problem 3.30. Find the following limits and give a justification (which can just be quoting the right proposition or theorem) for your answer.

(a) $\lim_{n\to\infty} \dfrac{4n^3 + 5n^6}{7n^3 - 8n + 7}$

(b) $\lim_{n\to\infty} \dfrac{-3n^2 + 1}{7n^5 - 19}$ $\square$

Problem 3.31. Let $0 < a < 1$. Give a ε, N proof that

$$\lim_{n\to\infty} a^k = 0.$$

Hint: Let $\varepsilon > 0$. We have proven that for $a \in (0,1)$ there is a natural number N with $a^N < \varepsilon$. $\square$

Problem 3.32. Let $a > 0$ and $x \geq a/4$. Show

$$|\sqrt{x} - \sqrt{a}| \leq \frac{2|x-a|}{3\sqrt{a}}.$$

$\square$

Proposition 3.36. *If $\langle a_n \rangle_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$, say*

$$\lim_{n\to\infty} a_n = a$$

with $a > 0$. Then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$$

Problem 3.33. Give a N, ε proof of this. $\square$

3.2. **Using limits to show sets are closed.** The next few definitions, propositions, and problems are practice in using the definitions.

Definition 3.37. Let E be a metric space and S a subset of E. Then $p \in E$ is an ***adherent point*** of s if and only if every open ball about p contains at least one points of S.

Problem 3.34. To get a feel for what this means, do the following

(a) Show that every point of S is an adherent point of S.
(b) What are the adherent points of the open interval $(0,1)$?
(c) What are the adherent points of the rational numbers, $\mathbb{Q}$, in the real numbers $\mathbb{R}$? $\square$

Theorem 3.38. *A set is closed if and only if it contains all its adherent points.*

Problem 3.35. Prove this. *Hint:* Let S be a subset of the metric space E.

(a) First show that if S is closed that it contains all its adherent points. So assume S is closed and p is an adherent point of S. Towards a contradiction assume that p is not in F. Then $p \in \mathcal{C}(S)$, the compliment of S. As S is closed, the set $\mathcal{C}(S)$ is open. Therefore there is an $r > 0$ such that $B(a, r) \subseteq \mathcal{C}(S)$. Show that this contradicts that p is an adherent point of S.

(b) Now show that if S is not closed, then S has an adherent point, p, with $p \notin S$. For if S is not closed, $\mathcal{C}(S)$ is not open and therefore $\mathcal{C}(S)$ has a point $p \in \mathcal{C}(S)$ such that no open ball about p is is contained in $\mathcal{C}(S)$. Show that p is an adherent point of S. $\square$

In what follows we will often want to show that some set is closed. The following gives method for going this that works well in 87.3% of known proofs.

Theorem 3.39. *Let S be a subset of the metric space E. Then the following are equivalent.*

(a) S is closed.

(b) S contains the limits of its sequences in the sense that if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converges, say $x = \lim_{n\to\infty}$, then $x \in S$.

Remark 3.40. In practice it is the implication (b) $\implies$ (a) that is useful is showing that sets are closed. $\square$

Lemma 3.41. *Let S be a set in a metric space and p an adherent point of S. Then there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty}$ from S that converges to p.*

Problem 3.36. Prove this. *Hint:* Let p be an adherent point of S. This means that for every $r > 0$ the ball $B(p, r)$ contains a point of S. For each positive integer n let $p_n \in S$ be point of S that is in the ball $B(p, 1/n)$. Now show that $\lim_{n\to\infty} p_n = p$. $\square$

The converse of the last lemma is also true.

Lemma 3.42. *Let S be a set in a metric space and p a point that is a limit of a sequence of points from S. Then p is an adherent point of S.*

Problem 3.37. Prove this. *Hint:* Let S be a set in the metric space E and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of points from S that converges to the point $p \in E$. You need to show that p is an adherent point of S. That is if $r > 0$ the ball $B(a, r)$ contains a point of S. As $\lim_{n\to\infty} p_n = p$ we can use $\varepsilon = r$ in the definition of limit to see there is a $N > 0$ such that $n > N$ implies that $d(p_n, p) < r$. $\square$

Problem 3.38. Prove Theorem 3.39. *Hint:*

(a) $\implies$ (b). Assume that S is closed and that $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converge to the point p. Use some of the lemmas above to show that p is an adherent point of S and then use that closed sets contain their adherent points. $\square$

(b) $\implies$ (a). Assume that (b) holds. We wish to show that S is closed. It is enough to show that S contains all its adherent points. Let p be an adherent point of S. Then use one or more of the lemmas above to show that p is a limit of a sequence form S. $\square$

Problem 3.39. This is an example of using Theorem 3.39 to to a set is closed. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial. Then we have seen (Theorem 3.33) that if $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} f(p_n) = f(p)$. Let F be a closed subset of $\mathbb{R}$ and f a polynomial. Show that

$$S := f^{-1}[F] = \{x : f(x) \in F\}$$

is a closed subset of $\mathbb{R}$. *Hint:* Use Theorem 3.39. Let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence of points from S with $\lim_{n\to\infty} p_n = p$. To show that S is closed we need to show that $p \in S$. All we need to to is show that $p \in S$. By the definition of S we have $f(p_n) \in F$. Also we have $f(p) = \lim_{n\to\infty} f(p_n)$. Use Theorem 3.39 to show that $f(p) \in F$, and therefore $p \in f^{-1}[F]$. Now use Theorem 3.39 again to conclude that $S = f^{-1}[F]$ is closed. $\square$

3.3. Cauchy sequences, definition of completeness of metric spaces.

The following is one of the basic ideas in analysis.

Definition 3.43. Let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence in a metric space E. Then the sequence is a ***Cauchy sequence*** if and only if for all $\varepsilon > 0$, there is a $N > 0$ such that $m, n > N$ implies $d(p_m, p_n) < \varepsilon$. $\square$

A brief version would be that $\langle p_n\rangle_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall\varepsilon > 0\ \exists N > 0\big[\, m, n > N \quad\implies\quad d(p_m, p_n) < \varepsilon\,\big].$$

Proposition 3.44. *Every convergent sequence is a Cauchy sequence.*

Problem 3.40. Prove this. *Hint:* Let $\langle p_n\rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space and let p be its limit. Let N be so that

$$n > N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2}.$$

Then show that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

The converse is not true. There are Cauchy sequences that are not convergent.

Problem 3.41. Let $E = (0, 1)$ be the open unit interval with metric $d(x, y) = |x - y|$. Then show that the sequence $\langle 1/n\rangle_{n=1}^{\infty}$ is a Cauchy sequence that is not convergent to any point of E. $\square$

You may feel that the example of the last problem is a bit of a cheat as the sequence does converge in the larger space of all real numbers. And is some sense this is true, given a metric space, E, there is a natural way to expand it to a somewhat larger space that contains the limits of all Cauchy sequences from E.

Definition 3.45. Let $\langle p_n\rangle_{n=1}^{\infty}$. Then a ***subsequence*** of this sequence is a sequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$ where $n_1, n_2, n_3, \ldots$ are positive integers satisfying $n_1 < n_2 < n_3 < \cdots$.

Put a little differently $\langle p_{n_k}\rangle_{k=1}^{\infty}$ is a subsequence of $\langle p_n\rangle_{n=1}^{\infty}$ if and only if $\langle n_k\rangle_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers.

The following lemma is one of those things that is more or less obvious, but I include a proof for completeness.

Lemma 3.46. *Let $\langle n_k \rangle_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers, that is $n_1 < n_2 < n_3 < \cdots$ and in general $n_k < n_{k+1}$. Then $n_k \geq k$ for all k.*

Proof. We use induction on k. As n_1 is a positive integer we have $n_1 \geq 1$. This is the base of the induction. Assume that $n_k \geq k$. Then, as we are working with integers, $n_{k+1} > n_k$ implies $n_{k+1} \geq n_k + 1$ and this

$$n_{k+1} \geq n_k + 1 \geq k + 1$$

which closes the induction and completes the proof. $\square$

Proposition 3.47. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E, such that some subsequence of $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.*

Problem 3.42. Prove this. *Hint:* Let $\varepsilon > 0$. As the sequence is Cauchy, there is a N such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \frac{\varepsilon}{2}.$$

Let $n > N$, then for any k we have by the triangle inequality that

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p).$$

Now show that it is possible to choose k such that both $d(p_n, p_{n_k})$ and $d(p_{n_k}, p)$ are less than $\varepsilon/2$. $\square$

Proposition 3.48. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space E. Let $\langle p_{n_k} \rangle_{k=1}^{\infty}$ be a subsequence of this sequence. Then $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is also convergent and has the same limit at the original sequence.*

Problem 3.43. Prove this. *Hint:* For all k we have $n_k \geq k$. $\square$

Definition 3.49. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Then this sequence is ***monotone increasing*** if and only if $x_n \leq x_{n+1}$ for all n. It is ***monotone decreasing*** if and only if $x_n \geq x_{n+1}$ for all n. It is ***monotone*** if it is either monotone increasing or monotone decreasing. $\square$

Theorem 3.50. *A bounded monotone sequence in $\mathbb{R}$ is convergent.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a bounded monotone sequence. We first assume that it is monotone increasing. Let

$$S = \{x_n : n = 1, 2, \ldots\}$$

be the set of values of the sequence. As the sequence is bounded, this set is bounded. Therefore, by Least Upper bound Axiom, this set has a least upper bound $b = \sup(S)$. We now show that the sequence converges to b.

Let $\varepsilon > 0$. Then $b - \varepsilon < b$ and b is the least upper bound of S, therefore $b - \varepsilon$ is not an upper bound for S. Whence there is positive integer N such that $b - \varepsilon < x_N$. Then for any $n > N$ we have

$$\begin{aligned} b - \varepsilon &< x_N \\ &\le x_n \qquad (x_N \le x_n \text{ as the sequence is monotone increasing.}) \\ &\le b \qquad (\text{as } b \text{ is an upper bound for } S \text{ and } x_n \in S.) \end{aligned}$$

Therefore we have $b - \varepsilon < x_n \le b$ for all $n > N$. Thus $n > N$ implies $|x_n - b| < \varepsilon$ and thus $\lim_{n\to\infty} x_n = b$. □ □

Problem 3.44. Modify the last proof so show that if $\langle x_n \rangle_{n=1}^{\infty}$ is bounded and monotone decreasing that it converges to $\inf\{x_n : n = 1, 2, 3, \ldots\}$. □

Theorem 3.51. *Every sequence of real numbers has a monotone subsequence.*

Problem 3.45. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Call x_n a ***peak point*** if $x_n \ge x_m$ for all $m > n$. (That is x_n is greater than or equal to all the values that follow it.)

Case 1: *There are infinitely many peak points.* In this case there is an infinite subsequence of the sequence consisting of peak points. Show this subsequence is monotone decreasing.

Case 2: *There are only finitely many peak points.* Let N be the largest n such that x_n is a peak point. Thus if $n > N$ the point x_n is not a peak point and therefore there is $m > n$ with $x_n > x_n$. Let $n_1 = N_1$. Then $n_1 > N$ and so there is a $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. But then $n_2 > N$ and thus there is $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. Continue in this manner to show that there is an infinite increasing subsequence. □

Proposition 3.52. *Let E be a metric space. Then every Cauchy sequence in E is bounded. (That is the sequence is contained in some ball.)*

Problem 3.46. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Let $\varepsilon = 1$ (or any other positive number that you like). Then there is $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon = 1.$$

Let $a = x_{N+1}$ and set

$$r = 1 + \max\{1, d(a, x_1), d(a, x_2), \ldots, d(a, x_N)\}.$$

Then show that $p_n \in B(a, r)$ for all n. □

Theorem 3.53. *Every Cauchy sequence in $\mathbb{R}$ converges.*

Problem 3.47. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Then by Proposition 3.52 this sequence is bounded. By Theorem 3.51 this sequence has a monotone subsequence. By Theorem 3.50 this monotone subsequence converges. Put these facts together with Proposition 3.47 to prove that the sequence $\langle x_n \rangle_{n=1}^{\infty}$ converges. □

This property of a metric space, that Cauchy implies convergent, is important enough to give a name.

Definition 3.54. The metric space E is ***complete*** if and only if every Cauchy sequence in E converges. $\square$

So we can restate Theorem 3.53 as

Proposition 3.55. *The real numbers, $\mathbb{R}$, with their usual metric is a complete metric space.* $\square$

3.4. **Completeness of $\mathbb{R}^n$.** We can get more examples by looking at closed subsets of complete metric spaces.

Proposition 3.56. *Let E be a metric space and F a closed subset of E. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence of points of F that converges in E to some point p. Then $p \in F$. (A nice restatement of this is that a closed set contains all its limit points.)*

Problem 3.48. Prove this. *Hint:* Towards a contradiction assume that $p \notin F$. Then as F is closed, the compliment $\mathcal{C}(F)$ is open. As $p \in \mathcal{C}F$ by the definition an open set, there is a $r > 0$ such that $B(p, r) \subseteq \mathcal{C}(F)$. But $\lim_{n\to\infty} p_n = p$ and therefore if we let $\varepsilon = r$ there is a $N > 0$ such that $n > 0$ implies $d(p_n, p) < \varepsilon = r$. This this leads to a contradiction. $\square$

Proposition 3.57. *Let E be a complete metric space and F a closed subset of E. Then F, considered as a metric space in its own right, is complete.*

Problem 3.49. Prove this. *Hint:* Let $\langle p_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence from F. As E is complete this sequence converges to some point, p, of E. To finish the proof it is enough to show that $p \in F$. $\square$

Recall that we have made $\mathbb{R}^n$ into metric spaces with the metric

$$d(p, q) := \|p - q\|$$

where

$$\|p\| = \|(p_1, p_2, \ldots, p_n)\| = \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}.$$

Theorem 3.58. *With this metric $\mathbb{R}^n$ is complete.*

Problem 3.50. Prove this in the case of $n = 3$. *Hint:* Here is the proof for $n = 2$. Let $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^2$. Note we have the inequality

$$|x_m - x_n| = \sqrt{(x_n - x_n)^2} \le \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} = d(p_m, p_n)$$

with a similar calculation showing

$$|y_m - y_n| \le d(p_m, p_n).$$

As $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ is Cauchy there is a $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

From the inequalities above this gives

$$m, n > N \quad \text{implies} \quad |x_m - x_n|, |y_m - y_n| \le \|p_m - p_n\| < \varepsilon.$$

Therefore both of the sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ are Cauchy and as $\mathbb{R}$ is complete this implies that they both converge. Thus there are $x, y \in \mathbb{R}^n$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

and thus there are $N_1 > 0$ and $N_2 > 0$ such that

$$n > N_1 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{\sqrt{2}}$$
$$n > N_2 \quad \text{implies} \quad |y_n - y| < \frac{\varepsilon}{\sqrt{2}}$$

Then if $N = \max\{N_1, N_2\}$ and $p = (x, y)$

$$\begin{aligned} n > N \quad \text{implies} \quad \|p_n - p\| &= \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &< \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} \\ &= \varepsilon. \end{aligned}$$

which shows that the Cauchy sequence $\langle p_n\rangle_{n=1}^\infty = \langle (x_n, y_n)\rangle_{n=1}^\infty$ has the limit (x, y). As this was an arbitrary Cauchy in $\mathbb{R}^2$, this shows $\mathbb{R}^2$ is complete.

Now you do the proof for $\mathbb{R}^3$. □

3.5. **Sequential compactness and the Bolzano-Weierstrass Theorem.**

Definition 3.59. A subset S of a metric space is ***sequentially compact*** if and only if every sequence $\langle p_n\rangle_{n=1}^\infty$ of points from S has a subsequence that converges to a point of S. □

Problem 3.51. Let S be a finite subset of a metric space. Then S is sequentially compact. *Hint:* Let $S = \{s_1, s_2, \ldots, s_m\}$. For each $j \in \{1, 2, \ldots, m\}$ let $\mathcal{N}_j = \{n : p_n = s_j\}$. As the union of the sets $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_m$ it the infinite set $\mathbb{N} = \{1, 2, 3, \ldots\}$ at least one of them is infinite. Say that $\mathcal{N}_j$ is infinite, $\mathcal{N}_j = \{n_1, n_2, n_3, \ldots\}$ and consider the subsequence $\langle p_{n_k}\rangle_{k=1}^\infty$. □

Theorem 3.60 (Bolzano–Weierstrass Theorem)**.** *Every closed bounded subset of* $\mathbb{R}$ *is sequentially compact.*

Problem 3.52. Prove this. *Hint:* Let S be a closed bounded subset of $\mathbb{R}$ and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence of points from S. Then the sequence is bounded (because S is bounded). Also (Theorem 3.51) this sequence has a monotone subsequence. At some point in finishing the proof you will need to use Proposition 3.56. □

Corollary 3.61 (Bolzano–Weierstrass Theorem for sequences)**.** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Proof. Let $\langle x_n\rangle_{n=1}^\infty$ be bounded sequence in $\mathbb{R}$. As the sequence is bounded there is a closed ball $\overline{B}(a, r) = [a - r, a + r]$ that contains $\langle x_n\rangle_{n=1}^\infty$. The set

$\overline{B}(a,r)$ is a closed bounded subset of $\mathbb{R}$ and so by the Bolzano-Weierstrass the $\langle x_n\rangle_{n=1}^{\infty}\langle x_n\rangle_{n=1}^{\infty}$ has a convergent subsequence. □

Theorem 3.62 (General Bolzano–Weierstrass Theorem)**.** *Every closed bounded subset of $\mathbb{R}^n$ is sequentially compact.*

Lemma 3.63. *Let $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n)\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^2$. Then this sequence converges if and only if both the sequences*

$$\langle x_n\rangle_{n=1}^{\infty} \quad \textit{and} \quad \langle y_n\rangle_{n=1}^{\infty}$$

converge.

Proof. Let $p = (x, y) \in \mathbb{R}^2$. Then, as we saw in Problem 3.50 the inequalities

$$|x_n - x|,\ |y_n - y| \le \sqrt{(x - x_n)^2 + (y_n - y)^2} = d(p_n, p)$$

Therefore if $\langle p_n\rangle_{n=1}^{\infty}$ converges, as $\lim_{n\to\infty} p_n = p$ then for any $\varepsilon > 0$ there is a $N > 0$ such that $n > N$ implies $d(p_n, p) < \varepsilon$. Therefore for this N we have

$$n > N \quad \text{implies} \quad |x_n - x|, |y_n - y| \le d(p_n, p) < \varepsilon$$

and therefore we have $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Conversely assume that both the limits $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist, say $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Therefore there are N_1 and N_2 such that

$$n > N_1 \quad \text{implies} \quad |x - x_n| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and}$$

Thus is $N = \max\{N_1, N_2\}$ we have, just as in Problem 3.50,

$$n > N \quad \text{implies} \quad d(p_n, p) < \varepsilon$$

which shows that $\langle p_n\rangle_{n=1}^{\infty}$ converges. Now show the limit is in S. □

Lemma 3.64. *Let $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n, z_n)\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^3$. Then this sequence converges if and only if all three of they sequences*

$$\lim_{n\to\infty} x_n, \quad \lim_{n\to\infty} y_n, \quad \textit{and} \quad \lim_{n\to\infty} z_n$$

converge.

Problem 3.53. Prove Theorem 3.62 for $n = 3$. *Hint:* Here is the proof for $n = 2$. Let S be a closed bouneded subset of $\mathbb{R}^n$ and $\langle p_n\rangle_{n=1}^{\infty} = \langle (x_n, y_n\rangle_{n=1}^{\infty}$ a sequence in S. As S is bounded the sequence $\langle p_n\rangle_{n=1}^{\infty}$ is bounded. But

$$|x_n|,\ |y_n| \le \sqrt{|x_n|^2 + |y_n|^2} = d(\vec{0}, p_n)$$

and therefore both of the sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ are bounded. As the sequence $\langle x_n\rangle_{n=1}^{\infty}$ is bounded by Corollary 3.61 it has a convergent subsequence $\langle x_{n_k}\rangle_{k=1}^{\infty}$. The sequence $\langle y_{n_k}\rangle_{k=1}^{\infty}$ is a subsequence of the bounded sequence $\langle y_n\rangle_{n=1}^{\infty}$ and therefore $\langle y_{n_k}\rangle_{k=1}^{\infty}$ is also bounded. Therefore we can use Corollary 3.61 again to get a subsequence $\langle y_{n_{k_j}}\rangle_{j=1}^{\infty}$ of the subsequence $\langle y_{n_k}\rangle_{k=1}^{\infty}$ $\langle y_{n_k}\rangle_{k=1}^{\infty}$ such that $\langle y_{n_{k_j}}\rangle_{j=1}^{\infty}$ converges. Now note that the subsequence $\langle x_{n_{k_j}}\rangle_{j=1}^{\infty}$ is a convergent subsequence of the convergent sequence

$\langle x_{n_k}\rangle_{k=1}^\infty$. But a subsequence of a convergent subsequence is convergent (Proposition 3.48) and therefore $lax_{n_{k_j}}\rangle_{j=1}^\infty$ is convergent. But then both of the sequences

$$\langle x_{n_{k_j}}\rangle_{j=1}^\infty \quad \text{and} \quad \langle y_{n_{k_j}}\rangle_{j=1}^\infty$$

converge and therefore by Lemma 3.63 this implies the sequence

$$\langle p_{n_{k_j}}\rangle_{j=1}^\infty = \langle (x_{n_{k_j}}, y_{n_{k_j}})\rangle_{j=1}^\infty$$

converges. Let $p = \lim_{n\to\infty} p_{n_{k_j}}$. Then, as S is closed, Proposition 3.56 implies $p \in S$. As $\langle p_n\rangle_{n=1}^\infty$ was any sequence from the closed bounded set, S, this shows that every sequence from a closed bounded subset of $\mathbb{R}^2$ has a subsequence that converges to a point of S. Therefore closed bounded subsets of $\mathbb{R}^n$ are sequentially compact. □

Corollary 3.65 (General Bolzano-Weierstrass for sequences)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Problem 3.54. Prove this. *Hint:* See the proof of Corollary 3.61. □

Proposition 3.66. *Every sequentially compact subset of a metric space is closed and bounded.*

Problem 3.55. Prove this. *Hint:* Let S be sequentially compact in E. First show that S is bounded. Towards a contradiction assume that it is not bounded. Let q be any point of S. Because S is not bounded there for each positive integer n there is a point $p_n \in S$ with $d(q, p_n) > n$. The set S is sequentially compact and therefore the sequence $\langle p_n\rangle_{n=1}^\infty$ has a convergent subsequence $\langle p_{n_k}\rangle_{k=1}^\infty$. Let $p = \lim_{k\to\infty} p_{n_k}$. Then using $\varepsilon = 1$ in the definition of limit we have that there is a $K > 0$ such that $k > K$ implies that $d(p, p_{n_k}) < 1$. Whence for all $k > K$ we have by the triangle inequality

$$n_k < d(q, p_{n_k}) \le d(q, p) + d(p, p_{n_k}) < d(q, p) + 1$$

which gives a contradiction (why?)

Now use sequential compactness to show S is closed. One way is to show that sequential compactness implies that S contains all its adherent points. □

Remark 3.67. We have seen, Theorem 3.62, that every closed bounded of $\mathbb{R}^n$ is sequentially compact. And the last proposition shows that a sequentially compact subset is closed and bounded. But it is important to realize that not all closed bounded subsets of all subsets of all metric spaces are sequentially compact. The next problem give an example. □

Problem 3.56. Let $E = (0, \infty)$ and let $S = (0, 1]$. Here we are using the metric $d(x, y) = |x - y|$. Show that S is a closed bounded subset of E, but that S is not sequentially compact. □

3.6. **Open covers and the Lebesgue covering lemma.** We recall a bit of set theory. Let E a set and $\mathcal{U}$ a collection of subsets of E. (At bit more formally if $U \in \mathcal{U}$ then $U \subseteq E$.) The ***union*** of $\mathcal{U}$ is

$$\bigcup \mathcal{U} = \{x : x \in U \text{ for at least one } U \in \mathcal{U}\}.$$

We will sometimes use the notation

$$\bigcup_{U \in \mathcal{U}} U$$

or some trivial variants of this notation. For example

$$\bigcup_{n=1}^{\infty} (-n, n) = (-\infty, \infty)$$

or

$$\bigcup_{x \in [0,1]} (x-1, x+1) = (-1, 2).$$

Of course there is the ***intersection*** of $\mathcal{U}$ which it

$$\bigcap \mathcal{U} = \{x : x \in U \text{ for all } U \in \mathcal{U}\}.$$

which can also be written as

$$\bigcap_{U \in \mathcal{U}} U$$

with such variants as

$$\bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n) = [a, b]$$

(which gives anther example of an infinite intersection of open sets not being open), and

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \varnothing.$$

Definition 3.68. Let E be a metric space and $S \subseteq E$. Then $\mathcal{U}$ is an ***open cover*** of S if and only if the following hold

(a) Each element, U, of $\mathcal{U}$ is an open subset of E.
(b) $S \subseteq \bigcup \mathcal{U}$. $\square$

Anther way to say $S \subseteq \cup \mathcal{U}$ is that for all $x \in S$ there is an $U \in \mathcal{U}$ with $x \in U$. This is nothing more than a restatement of the definition of the union, but in practice is how we often work with open covers.

Theorem 3.69 (Lebesgue Covering Theorem)**.** *Let S be a sequentially compact subset of the metric space E and let $\mathcal{U}$ be an open over of S. Then there is a $r > 0$ (often called a* ***Lebesgue number*** *of the cover) such that for all $x \in S$ there are is a $U \in \mathcal{U}$ with $B(x, r) \subseteq U$.*

A restatement is that given an open cover $\mathcal{U}$ of a sequentially compact set S there is a $r > 0$ (which depends on both S and $\mathcal{U}$) such that every point of S is contained in a ball of radius r that is contained in some open set $U \in \mathcal{U}$. In practice this means that in working with open covers, we can sometimes replace them with a cover by balls all with the same radius.

Problem 3.57. Prove Theorem 3.69. *Hint:* Towards a contradiction assume that there is an open cover $\mathcal{U}$ of a sequentially compact set S where the Lebesgue Covering Theorem does not hold. This means that for all $r > 0$ there is a point $x \in S$ such that the ball $B(x, r)$ is not contained in any $U \in \mathcal{U}$.

For each positive integer n let $x_n \in S$ be a point where the ball $B(x_n, 1/n)$ is not contained in any of the sets $U \in \mathcal{U}$. As S is sequentially compact, the sequence $\langle x_n \rangle_{n=1}^{\infty}$ has a convergent subsequence, $\langle x_{n_k} \rangle_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} = x$ where $x \in S$. As $x \in S$ and $\mathcal{U}$ is an open cover of S there is some $U \in \mathcal{U}$ with $x \in U$. As U is open there is a $r > 0$ such that $B(x, r) \subseteq U$. Because $\lim_{k\to\infty} x_{n_k} = x$ there is a $N > 0$ such that

$$k > N \quad \text{implies} \quad d(x_{n_k}, x) < \frac{r}{2}.$$

Now show that if we choose k such that both $k > N$ and $1/n_k < r/2$ hold then

$$B(x_{n_k}, 1/n_k) \subseteq B(x, r) \subseteq U$$

and explain why this leads to a contradiction. □

3.7. Open covers and compactness.

Definition 3.70. Let S be a subset of the metric space E. Then S is ***compact*** if and only if every open cover of S has a finite subcover. Explicitly the means that if $\mathcal{U}$ is an open cover of S then there is a finite set $\{U_1, U_2, \ldots, U_m\} \subseteq \mathcal{U}$ with

$$S \subseteq U_1 \cup U_2 \cup \cdots \cup U_m$$

□

Theorem 3.71. *Every sequentially compact set in a metric space is compact.*

Problem 3.58. Prove this. *Hint:* Towards a contradiction assume that S is a sequentially compact subset of some the metric space E that is not compact. That is there is some open cover of $\mathcal{U}$ of S that as no finite subcover. Let r be a Lebesgue number for this open cover. That is for every $p \in S$ there is some $U \in \mathcal{U}$ such that $B(p, r) \subseteq U$. We know that such an r exists by the Lebesgue Covering Theorem 3.69. Define a sequence of points $p_1, p_2, p_3, \ldots \in S$ and a sequence $U_1, U_2, U_3, \ldots \in \mathcal{U}$ follows. Let p_1 be any element of S. Then there is a $U_1 \in \mathcal{U}$ such that $B(p_1, r) \subseteq U_1$. Now assume that $p_1, 2_2, \ldots, p_n \in S$ and $U_1, U_2, \ldots, U_n \in \mathcal{U}$ have been defined such that

$$p_j \notin U_1 \cup U_2 \cup \cdots \cup U_{j-1} \quad \text{and} \quad B(x_j, r) \subseteq U_j$$

for $j = 1, 2, \ldots, n$. Now

(a) Explain why there is an $p_{n+1} \in S$ such that $p_{n+1} \notin U_1 \cup U_1 \cup \cdots U_n$.
(b) There is a $U_{n+1} \in \mathcal{U}$ such that $B(p_{n+1}, r) \subseteq U_{n+1}$.

Finish the proof by showing that if $m \neq n$, say $m < n$, then $p_n \notin U_m$ and $B(p_m, r) \subseteq U_m$ implies that $d(p_m, p_n) \geq r$ and thus the sequence $\langle p_n \rangle_{n=1}^{\infty}$ has no convergent subsequence (if $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence has $d(p_{x_k}, p_{x_\ell}) \geq r$ for $k \neq \ell$. Use this to show the subsequence is not Cauchy) which contradicts that S is sequentially compact. □

The converse of the last Theorem is also true.

Theorem 3.72. *Every compact set in a metric space is sequentially compact.*

Lemma 3.73. *Let S be a compact set in a metric space and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then there is a point $p \in S$ such that for all $r > 0$ the set*

$$\{n : p_n \in B(p, r)\}$$

is infinite.

Problem 3.59. Prove this. *Hint:* This is a very typical use of compactness in a proof. Towards a contradiction assume that there is a compact set S where this does not hold. Then for each $x \in S$ there is a $r_x > 0$ such that $\{n : p_n \in B(x, r_x)\}$ is finite. Set

$$\mathcal{U} = \{B(x, r_x) : x \in S\}.$$

Show that $\mathcal{U}$ is an open cover of S. By compactness there is a finite subcover, say that

$$S \subseteq B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \cdots \cup B(x_m, r_{x_m}).$$

Now for each natural number $n \in \mathbb{N}$ we have that $p_n \in B(x_j, r_{x_j})$ for at least one $j \in \{1, 2, \ldots, m\}$. Thus, by the pigeon hole principle, there is at least one j where the set $\{n : p_n \in B(x_j, r_{x_j})\}$ is infinite. Explain why this is a contradiction. □

Problem 3.60. Prove this. *Hint:* Let S be a compact set in a metric space E. We wish to show that every sequence $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence converging to a point of S. Towards this end let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in S. By Lemma 3.73 there is a point $p \in S$ such that for all $r > 0$ the set of $n \in \mathbb{N}$ with $p_n \in B(x, r)$ is infinite. Show this implies $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence converging to p. □

3.8. **Connected sets.** We can to describe sets that are connected in the everyday sense that they do not split into smaller pieces. Here is the precise definition.

Definition 3.74. The metric space is ***connected*** if and only if E is not the disjoint union of two nonempty open sets. □

To be very explicit this means there are no open sets A and B with

$$E = A \cup B$$
$$A \cap B = \varnothing$$
$$A \neq \varnothing$$
$$B \neq \varnothing$$

If a metric space is not connected it is ***disconnected***. Let E be disconnected, then there are open sets A and B, both nonempty with

$$E = A \cup B \quad \text{and} \quad A \cap B = \varnothing. \tag{6}$$

It is convenient to have a name for such a splitting of disconnected space. If E is disconnected, then a ***disconnection*** of E is a pair of nonempty open sets A and B such that (6) holds.

Proposition 3.75. *If $E = A \cup B$ is a disconnection of the metric space E, then the sets A and B are both open and closed.*

Problem 3.61. Prove this. *Hint:* Note that $E = A \cap B$ and $A \cap B = \varnothing$ together imply that $\mathcal{C}(A) = B$ (the complement of A in E is B) and $\mathcal{C}(B) = A$. The sets A and B are open by the definition of disconnection. But then $A = (B)$ and $B = (A)$ and the compliment of an open set is closed. □

Call a set in a metric space ***clopen*** if and only if it is both open and closed.

Proposition 3.76. *Let E be a metric space. Then the following are equivalent.*

(a) E is connected.
(b) There is no decomposition $E = A \cup B$ with A and B both closed, $A \neq \varnothing$, $B \neq \varnothing$ and $A \cap B = \varnothing$.
(c) The only clopen sets in E are E and $\varnothing$.

Problem 3.62. Prove this. □

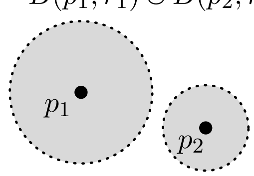
Let us look at some examples. First we will look at examples of disconnected sets.

Problem 3.63. Let X be a metric space, $p_1, p_2 \in X$ and $r_1, r_2 > 0$ with

$$r_1 + r_2 \leq d(p_1, p_2), Let$$

and set

$$E = B(p_1, r_1) \cup B(p_2, r_2).$$



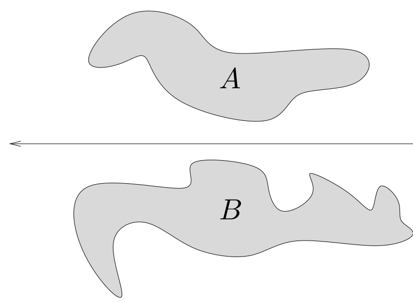
Show E is disconnected. *Hint:* As $r_1 + r_2 \leq d(p_1, p_2)$ the two balls are disjoint. And they are both open. □

Proposition 3.77. *Let $E \subseteq \mathbb{R}$. Say that E is split if there is a point $p \notin E$ such that there are $a, b \in E$ with $a < p < b$. Then that if E is split, it E is disconnected.*

Proof. Prove this. *Hint:* Let $A = (-\infty, p) \cap E$ and $B = E \cap (p, \infty)$. Then $a \in A$ and $b \in B$ so A and B are nonempty, Show that A and B are open subsets of E. $\square$ $\square$

Problem 3.64. Here is a two dimensional analogue of the last proposition. Let

$$U = \{(x, y) \in \mathbb{R}^2 : y > 0\} \qquad \text{(The upper half plane)}$$
$$V = \{(x, y) \in \mathbb{R}^2 : y < 0\} \qquad \text{(The lower half plane).}$$



Let A be a nonempty subset of U and B a nonempty subset of V and let $E = A \cup B$. Show that $E = A \cup B$ is a disconnection of E and therefore E is disconnected. $\square$

Lemma 3.78. *Let E be a metric space and $S \subseteq E$ a subset of E. Then S is metric space in its on right (using the distance function from E). Let $A \subseteq S$ be a subset of S. Then A is open in S if and only if there is an open set U in E such that $A = U \cap S$.*

Proof. For a point $a \in S$ let

$$B_E(a, r) = \{x \in E : d(x, a) < r\}$$
$$B_S(a, r) = \{x \in S : d(x, a) < r\}$$

be the open balls of radius r about a in the spaces E and S respectively. Then

$$B_S(a, r) = S \cap B_E(a, r).$$

First assume that $A = S \cap U$ where U is open in E and let $a \in A$. Then, by the definition of U being open in E, there is a $r > 0$ such that $B_E(a, r) \subseteq U$. But then

$$B_S(a, r) = S \cap B_E(a, r) \subseteq S \cap U = A.$$

Thus A contains the ball $B_S(a, r)$. As a was any point of A this shows that S contains a ball of S about any of its points and therefore A is open in S.

Conversely assume that A is open in S. We wish to find an open set U of E such that $A = S \cap U$. By the definition of A being open in S for each

$a \in A$ there is a $r_a > 0$ such that
$$B_S(a, a_r) \subseteq A.$$
This implies
$$\begin{aligned} A &= \bigcup_{a \in A} B_S(a, r_r) \\ &= \bigcup_{a \in A} (S \cap B_E(a, r_a)) \\ &= S \cap \bigcup_{a \in A} B_E(a, r_a) \\ &= S \cap U \end{aligned}$$
where
$$U = \bigcup_{a \in A} B_E(a, r_a)$$
is a union of open balls of E and therefore is an open set in E. $\square$

Proposition 3.79. *Let E be a metric space and for $\alpha \in I$ let S_α be a connected subset of E where I is some index set. Assume that for all $\alpha, \beta \in I$ that*
$$S_\alpha \cap S_\beta \neq \varnothing.$$
Then the union
$$S = \bigcup_{\alpha \in I} S_\alpha$$
is connected.

Proof. Towards a contradiction assume that S is not connected, and let
$$S = A \cup B$$
be a disconnection of S where A and B are open in S, each is nonempty and $A \cap B = \varnothing$. As A and B are nonempty there is are $a \in A$ and $b \in B$. As $S = \bigcup_{\alpha \in I} S_\alpha$, there are $\alpha, \beta \in I$ with $a \in S_\alpha$ and $b \in S_\beta$. Then $a \in A \cap S_\alpha$ and therefore $A \cap S_\alpha \neq \varnothing$. We now claim that $S_\alpha \cap B = \varnothing$. For if $S_\alpha \cap B \neq \varnothing$ then, as $S_\alpha \subseteq S = A \cup B$, implies
$$S_\alpha = (S_\alpha \cap A) \cup (S_\alpha \cap B). \tag{7}$$
The sets A and B are open in S, and therefore, by Lemma 3.78, the sets $A \cap S_\alpha$ and $S_\alpha \cap B$ are open in S_α. This implies that (7) is a disconnection of S_α, contradicting that S_α is connected. But $S_\alpha \subseteq A \cup B$ and $S_\alpha \cap B = \varnothing$ implies $S_\alpha \subseteq A$.

A similar argument shows that $S_\beta \subseteq B$. Whence
$$S_\alpha \cap S_\beta \subseteq A \cap B = \varnothing$$
contradicting that $S_\alpha \cap S_\beta \neq \varnothing$. $\square$

We have yet to get a nontrivial example of a connected space.

Theorem 3.80. *Let I be an interval in $\mathbb{R}$. Then I is connected.*

Proof. The proof will only use the following property of the interval I: if $a, b \in I$ with $a < b$, then $[a, b] \subseteq I$.

Now assume, towards a contradiction, that I is disconnected and let

$$I = A \cup B$$

be a disconnection of I. As A and B are nonempty there are elements $a \in A$ and $b \in B$. By relabeling if need be we can assume that $a < b$. Then $[a, b] \subseteq I$ and therefore $I = A \cup B$ implies

$$[a, b] = ([a, b] \cap A) \cup ([a, b] \cap B)$$

is a disconnection of $[a, b]$. To simplify notation we replace I by $[a, b]$ by $[a, b] \cap A$ by A and $[a, b] \cap B$ by B and assume that we have a disconnection

$$[a, b] = A \cup B \quad \text{with} \quad a \in A \quad \text{and} \quad b \in B.$$

By Proposition 3.75 the sets A and B are both closed in $[a, b]$.

The set A is a subset of the interval $[a, b]$ and therefore A is bounded. So by the least upper bound axiom A has a least upper bound.

$$\alpha = \sup(A).$$

As we are in $\mathbb{R}$ the ball $B(\alpha, r)$ is just the interval $(\alpha - r, \alpha + r)$. If For any $r > 0$ we must have $B(\alpha, r) \cap A \neq \varnothing$, for otherwise $a - r$ would be an upper bound for A, contradicting that α is the least upper bound. Therefore α is an adherent point of A. As A is closed this implies that $\alpha \in A$.

Also for all $r > 0$ we have $B(\alpha, \beta) \cap B \neq \varnothing$. For if $(\alpha - r, \alpha + r) \cap B = \varnothing$, then $(\alpha - r, \alpha + r) \cap [a, b] \subseteq A$, which implies that A contains points x with $x > \alpha$, contradicting that α is an upper bound for A. Thus α is an adherent point of B and B is closed. Thus $\alpha \in B$.

But then we have $\alpha \in A \cap B$, which contradicts $A \cap B = \varnothing$. $\square$

Proposition 3.81. *Let $\varnothing \neq S \subseteq \mathbb{R}$ have the property that if $a, b \in S$ and $a < b$, then $[a, b] \subseteq S$. Then S is an interval.*

Proof. Define α and β by

$$\alpha = \begin{cases} -\infty, & \text{if } S \text{ is not bounded below;} \\ \inf(S), & \text{if } S \text{ is bounded below.} \end{cases}$$

$$\beta = \begin{cases} +\infty, & \text{if } S \text{ is not bounded above;} \\ \sup(S), & \text{if } S \text{ is bounded above.} \end{cases}$$

Now the proof splits into a annoying number of cases.

Case 1. $\alpha = -\infty$ and $\beta = +\infty$. Let $x \in \mathbb{R}$. Then as S is not bounded either above or below there are $a, b \in S$ with $a < x < b$. Then $x \in [a, b] \subseteq S$. Thus every $x \in \mathbb{R}$ is in S and therefore $S = \mathbb{R}$ which is an interval.

Case 2. $\alpha = -\infty$ and $\beta < +\infty$. Let $x < \beta$. Then as S is not bounded below and $x < \sup(S)$ there are $a, b \in S$ with $a < x < b$. Thus $x \in [a, b] \subseteq S$. As this holds for all $x \in (-\infty, \beta)$ we have $(-\infty, \beta) \subseteq S$. As $\beta = \sup(S)$ we also have $S \subseteq (-\infty, \beta]$. That is $(-\infty, \beta) \subseteq S(-\infty, \beta]$. Therefore the only possibilities are

$$S = (-\infty, \beta) \quad \text{or} \quad S = (-\infty, \beta]$$

both of which are intervals.

Case 3. $-\infty < \alpha$ and $\beta = +\infty$. Then an easy variant of the argument used in Case 2 shows that $(\alpha, \infty) \subseteq S \subseteq [\alpha, \infty)$ and therefore

$$S = (\alpha, \infty) \quad \text{or} \quad S = [\alpha, \infty)$$

and again these are both intervals.

Case 4. $-\infty < \alpha$ and $\beta < +\infty$ and $\alpha < \beta$. Let $x \in (\alpha, \beta)$. Then, as $\alpha = \inf(S)$ and $\beta = \sup(S)$ there are $a, b \in S$ with $a < x < b$ and thus $x \in [a, b] \subseteq S$. Therefore $(\alpha, \beta) \subseteq S \subseteq [\alpha, \beta]$. There are then only four possibilities:

$$S = (\alpha, \beta), \quad S = (\alpha, \beta], \quad S = [\alpha, \beta), \quad S = [\alpha, \beta].$$

These are all intervals.

Case 5. $\alpha = \beta$. Then S is the one point set $S = \{\alpha\}$, which we view as the interval $[\alpha, \alpha]$. □

Theorem 3.82. *In $\mathbb{R}$ the connected sets are just the intervals (Where we view one element sets as intervals $[a, a] = \{a\}$).*

Problem 3.65. Combine Theorem 3.80 and its proof together with Proposition 3.81 to prove this. □

4. Continuous functions.

4.1. **Continuity of a function at a point and some examples.** We now start the last big topic we will cover this term, which is continuous maps between metric spaces.

Definition 4.1. Let E and E' be metric spaces and $f \colon E \to E'$ a function from E to E'. Let $p_0 \in E$. Then f is ***continuous*** at p_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $p \in E$

$$d(p, p_0) < \delta \quad \text{implies} \quad d(f(p), f(p_0)) < \varepsilon.$$

□

Example 4.2. Here is an example of showing something is continuous. Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = 3x + 5$$

Then f is continuous at every point of $\mathbb{R}$. To see this let $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Let $\delta = \varepsilon/3$. Then if $|x - x_0| < \delta$ we have

$$\begin{aligned} |f(x) - f(x_0)| &= |3x + 5 - (3x_0 + 5)| \\ &= |3(x - x_0)| \\ &= 3|x - x_0| \\ &< 3\delta \\ &= \varepsilon. \end{aligned}$$

Proposition 4.3. *Let E be a metric space and $f\colon E \to E$ the identity map, that is $f(p) = p$ for all $p \in E$. Then f is continuous at all points of E.*

Problem 4.1. Prove this. □

Problem 4.2. Let E be a metric space.

(a) Let $p, x_0, q \in E$ show that

$$|d(q, x_0) - d(p, x_0)| \le d(p, q).$$

(b) Let $x_0 \in E$ and define $f(p)$ to be the distance of p from x_0, that is $f(p) = d(p, x_0)$. Show that f is continuous at all points of E. *Hint:* Use part (a) to show $|f(p) - f(q)| \le d(p, q)$. □

Recall that a map $f\colon E \to E'$ between metric spaces is ***Lipschitz*** if and only if there is a constant $M \ge 0$ such that

$$d'(f(p), f(q)) \le M d(p, q)$$

for all $p, q \in E$.

Proposition 4.4. *Let $f\colon E \to E'$ be a Lipschitz map between metric space. Then f is continuous at all points of E.*

Problem 4.3. Prove this. *Hint:* Set $\delta = \dfrac{\varepsilon}{M}$. □

Recall that on $\mathbb{R}^n$ we have defined the inner product

$$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{n} a_j b_j$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. This was used to define the norm on $\mathbb{R}^n$ as

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

This in turn was used to define the distance function on $\mathbb{R}^n$ by

$$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|.$$

Also recall that we have the Cauchy-Schwartz inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\| \|\mathbf{b}\|.$$

Problem 4.4. Let $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Define the function $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

Show that f is continuous at all points of $\mathbb{R}^n$. *Hint:* Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ then show

$$f(\mathbf{p}) - f(\mathbf{q}) = \mathbf{a} \cdot (\mathbf{p} - \mathbf{q}).$$

Use the Cauchy-Schwartz inequality to show $|f(\mathbf{p}) - f(\mathbf{q})| \le \|\mathbf{a}\| \|\mathbf{p} - \mathbf{q}\|$ and therefore f is Lipschitz with Lipschitz constant $M = \|\mathbf{a}\|$. □

Problem 4.5. Define the functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$ and $g(x,y) = y$. Show that f and g are continuous. *Hint:* As the two proofs are the same, it is enough to show that f is continuous. Let $\mathbf{a} = (1,0)$ and $b = 0$ then $f(x,y) = (x,y)\cdot\mathbf{a} + b$ so one way to do this is to reduce it to the previous problem. □

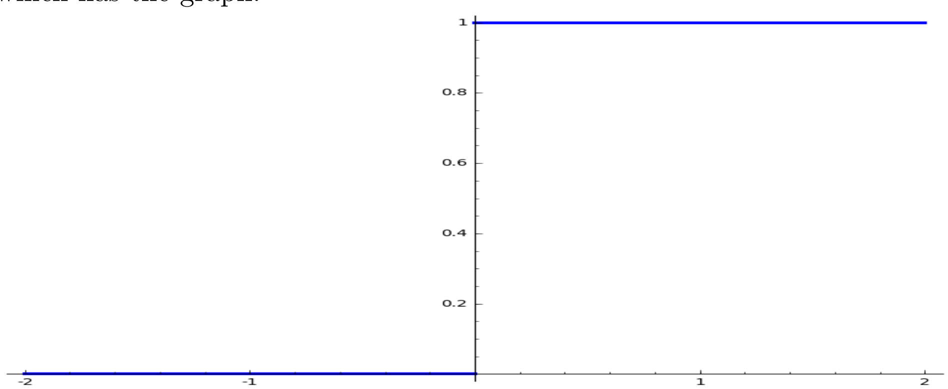
4.2. **Some examples of discontinuous functions.** We now give examples of some functions that are not continuous. We first record what it means for a function to not be continuous at a point.

Negation of Definition of Continuity. Let $f\colon E \to E'$ be a map between metric spaces. Let $p_0 \in E$. Then f is ***discontinuous*** at p_0 if and only if there is a $\varepsilon > 0$ such that for all $\delta > 0$ there is a $p \in E$ with $d(p, p_0) < \delta$ and $d'(f(p), f(p_0)) \geq \varepsilon$. □

We now look at the function

$$f(x) = \begin{cases} 0, & x \leq 0; \\ 1, & 0 < x. \end{cases}$$

which has the graph:



We now show this is discontinuous at $x = 0$. Let $\varepsilon = 1/2$. Then for any $\delta > 0$ there is an $x > 0$ with $0 < x < \delta$. Then $x > 0$ and so $f(x) = 1$. As $f(0) = 0$ we have $|f(x) - f(0)| = |1 - 0| = 1 > \varepsilon$ as required.

Here is a more exotic example.

Problem 4.6. Define a function by

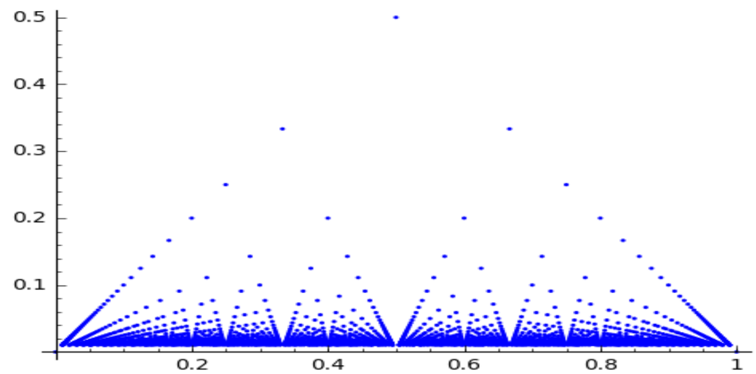
$$f(x) = \begin{cases} 0, & x \in \mathbb{Q}; \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

That is $f(x)$ is zero with x is a rational number, and $f(x)$ is one when x is irrational. Show that f is discontinuous at all points of $\mathbb{R}$.

Problem 4.7 (Optional). Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ is rational in lowest terms;} \\ 0, & x \text{ is irrational.} \end{cases}$$

Here is the graph for rationals in $(0,1)$ with denominators less than 100.



Show that f is continuous at all irrational points and discontinuous at all rational points. $\square$

4.3. **Sums, products, and quotients of real valued continuous functions are continuous.**

Problem 4.8. Let $f\colon (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \sqrt{x}$$

then show f is continuous at $x = 1$.

Solution: We first note that

$$|f(x)-f(1)| = |\sqrt{x}-1| = \left|\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)}\right| = \left|\frac{x-1}{\sqrt{x}+1}\right| \leq \left|\frac{x-1}{0+1}\right| = |x-1|.$$

Now let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then if $|x-1| < \delta$ implies

$$|f(x) - f(1)| \leq |x-1| < \delta = \varepsilon$$

which is just what is needed to show that $f(x)$ is continuous at $x = 1$. $\square$

Problem 4.9. Let $f\colon (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \sqrt{x}.$$

Show f is continuous at $x = a$ for any $a > 0$. $\square$

Theorem 4.5. *Let E be a metric space and $f, g\colon E \to \mathbb{R}$ be functions and $c_1, c_2 \in \mathbb{R}$ constants. Assume f and g are continuous at p_0. Then*

(a) $c_1 f + c_2 g$ is continuous at p_0.
(b) The product fg is continuous at p_0.
(c) If $g(p_0) \neq 0$, then quotient $\dfrac{f}{g}$ is continuous at p_0.

Problem 4.10. (a) Prove part (a) of the Theorem.

(b) Prove part (b) of the Theorem. *Hint:* Note that by our standard adding and subtracting trick

$$\begin{aligned}|f(p)g(p) - f(p_0)g(p_0)| &= |f(p)g(p) - f(p)g(p_0) + f(p)g(p_0) - f(p_0)g(p_0)| \\ &\le |f(p)||g(p) - g(p_0)| + |f(p) - f(p_0)||g(p_0)|\end{aligned}$$

By the continuity of f there is a $\delta_1 > 0$ such that

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |f(p) - f(p_0)| < 1.$$

Show

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |f(p)| < |f(p_0)| + 1.$$

Again by the continuity of f there is a $\delta_2 > 0$ such that

$$d(p, p_0) < \delta_2 \quad \text{implies} \quad |f(p) - f(p_0)| < \frac{\varepsilon}{2|g(p_0| + 1}.$$

The continuity of g gives us a $\delta_3 > 0$ such that

$$d(p, p_0) < \delta_3 \quad \text{implies} \quad |g(p) - g(p_0)| < \frac{\varepsilon}{2(|f(p_0)| + 1)}$$

Now set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and show

$$d(p, p_0) < \delta \quad \text{implies} \quad |f(p)g(p) - f(p_0)g(p_0)| < \varepsilon$$

□

Lemma 4.6. *Let E be a metric space and $g\colon E \to \mathbb{R}$ a function that is continuous at $p_0 \in E$ and with $g(p_0) \neq 0$. Then $\dfrac{1}{g}$ is also continuous at p_0.*

Problem 4.11. Prove this. *Hint:* As g is continuous at p_0 and $g(p_0) \neq 0$, there is a $\delta_1 > 0$ such that

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad |g(p) - g(p_0)| < \frac{|g(p_0)|}{2}.$$

Use this to show

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad \frac{1}{|g(p)|} < \frac{2}{|g(p_0)|},$$

and therefore

$$d(p, p_0) < \delta_1 \quad \text{implies} \quad \left|\frac{1}{g(p)} - \frac{1}{g(p_0)}\right| \le \frac{2|g(p_0) - g(p)|}{|g(p_0)|^2}$$

The continuity of g at p_0 implies there is a $\delta_2 > 0$ such that

$$d(p, p_0) < \delta_2 \quad \text{implies} \quad |g(p) - g(p_0)| < \frac{|g(p_0)|^2\varepsilon}{2}.$$

And you should be able to take it from here. □

Problem 4.12. Use Lemma 4.6 and part (b) of Theorem 4.5 to prove part (c) of Theorem 4.5. □

Proposition 4.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the polynomial*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then f is continuous at all points of $\mathbb{R}$.

Problem 4.13. Prove this. *Hint:* Probably the easiest way is by induction on n. The base of the induction is $n = 0$ in which case $f(x) = a_0$ is a constant which is clearly continuous. Or we can use the base case of $n = 1$ in which case $f(x) = a_1 x + a_0$ is Lipschitz and therefore continuous.

Here is what the induction step from $n = 4$ to $n = 5$ looks like. Assume that we know that all polynomials of degree 4 are continuous and let

$$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

be a polynomial of degree 5. Write it as

$$\begin{aligned} f(x) &= x(a_5x^4 + a_4x^3 + a_3x^2 + a_2x + a_1) + a_0 \\ &= xg(x) + a_0 \end{aligned}$$

where $g(x) = a_5x^4 + a_4x^3 + a_3x^2 + a_2x^1 + a_1$ is a polynomial of degree 4. By the induction hypothesis $g(x)$ is continuous and the function x is continuous. Whence f is of the form

$$f = (\text{continuous function}) \times (\text{continuous function}) + (\text{constant})$$

and therefore f is continuous. Use this idea to do the general induction step. $\square$

Theorem 4.8. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be maps between metric spaces. Let $p_0 \in E$ and assume that f is continuous at p_0 and g is continuous at $f(p_0)$. Then the composition $g \circ f$ is continuous at p_0.*

Problem 4.14. Prove this. *Hint:* Let $\varepsilon > 0$. By the definition of g being continuous at $f(p_0)$ there is a $\delta_1 > 0$ such that $d(q, f(p_0)) < \delta_1$ implies $d(g(q), g(f(p_0))) < \varepsilon$. By the definition of f being continuous at p_0 there is $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d(f(p), f(p_0)) < \delta_1$. Now show $d(p, p_0) < \delta$ implies $d\big((g \circ f)(p), (g \circ f)(p_0)\big) < \varepsilon$. $\square$

4.4. **Conditions equivalent to a function being continuous at a point.** We are going to give other conditions that imply a function is continuous, but first we review a bit of set theory from the beginning of the term.

Let $f\colon E \to E'$ be a map between sets. Recall that if $A \subseteq E$, then the ***image*** of A under f is

$$f[S] = \{f(x) : x \in A\}.$$

And if $B \subseteq E'$ the ***preimage*** of B under f is

$$f^{-1}[B] = \{x \in E : f(x) \in B\}.$$

We recall that taking preimages behaves well with respect to taking unions and intersections.

Proposition 4.9. *Let $f\colon E \to E'$ be a map between sets and let $\{S_\alpha\}_{\alpha\in I}$ be a collections of subsets of E'. (That is for each $\alpha \in A$ the $S_\alpha \subseteq E'$.) Then*

$$f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big] = \bigcup_{\alpha\in I} f^{-1}[S_\alpha] \qquad \textit{and}$$
$$f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] = \bigcap_{\alpha\in I} f^{-1}[S_\alpha],$$

Proof. To prove the first equality:

$$\begin{aligned} x \in f^{-1}\Big[\bigcup_{\alpha\in I} S_\alpha\Big] &\iff f(x) \in \bigcup_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for at least one } \alpha \in I \\ &\iff x \in f^{-1}[S_\alpha] \quad \text{for at least one } \alpha \in I \\ &\iff x \in \bigcup_{\alpha\in I} f^{-1}[S_\alpha]. \end{aligned}$$

This shows that $f^{-1}\Big(\bigcup_{\alpha\in I} S_\alpha\Big)$ and $\bigcup_{\alpha\in I} f^{-1}(S_\alpha)$ have the same elements and therefore are equal.

Likewise

$$\begin{aligned} x \in f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] &\iff f(x) \in \bigcap_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for all } \alpha \in I \\ &\iff x \in f^{-1}[S_\alpha] \quad \text{for all } \alpha \in I \\ &\iff x \in \bigcap_{\alpha\in I} f^{-1}[S_\alpha]. \end{aligned}$$

and therefore $f^{-1}\Big[\bigcap_{\alpha\in I} S_\alpha\Big] = \bigcap_{\alpha\in I} f^{-1}[S_\alpha]$. □

We recall that in the book's notation if S is a subset of some set E then the ***compliment*** of S in E is

$$\mathcal{C}(S) = \{x \in E : x \notin S\}.$$

That is $\mathcal{C}(S)$ is the set of points of E that are not in S. Taking compliments is also well behaved with respect to taking preimages.

Proposition 4.10. *Let $f\colon E \to E'$ be a map between sets and let $S \subseteq E'$. Then*

$$f^{-1}[\mathcal{C}(S)] = \mathcal{C}(f^{-1}[S]).$$

(Here $\mathcal{C}(S)$ is the compliment of S in E' and $\mathcal{C}(f^{-1}[S])$ is the compliment of $f^{-1}[S]$ in E.)

Proof. For $x \in E$ we have

$$\begin{aligned} x \in \mathcal{C}(f^{-1}[S]) &\iff x \in f^{-1}[S] \\ &\iff f(x) \notin S \\ &\iff f(x) \in \mathcal{C}(S) \\ &\iff x \in f^{-1}[\mathcal{C}(S)]. \end{aligned}$$

Therefore $\mathcal{C}(f^{-1}[S])$ and $f^{-1}[\mathcal{C}(S)]$ have the same elements and thus are equal. □

We summarize this as

Taking preimages preserves unions, intersections, and compliments.

Theorem 4.11. *Let $f\colon E \to E'$ be a function between metric spaces and let $p_0 \in E$. Then the following conditions are equivalent:*

(a) f is continuous at p_0.

(b) f does the right thing to sequences converging to p_0. Explicitly: If $p_1, p_2, p_3, \ldots$ is a sequence in E with $\lim_{n\to\infty} p_n = p_0$, then

$$\lim_{n\to\infty} f(p_n) = f(p_0).$$

(c) For any open ball $B(f(p_0), r)$ about $f(p_0)$, the preimage $f^{-1}\big[B(f(p_0), r)\big]$ contains a open ball about p_0. (That is there is a $\delta > 0$ such that $B(p_0, \delta) \subseteq f^{-1}\big[B(f(p_0), r)\big]$.)

Problem 4.15. In Theorem 4.11 prove (a) implies (b). *Hint:* Let $\varepsilon > 0$. Then the continuity of f at p_0 implies there is a $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d(f(p), f(p_0))$. Let $\lim_{n\to\infty} p_n = p_0$. Then there is a N such that $n \geq N$ implies $d(p_n, p_0) < \delta$. Now show that $n \geq N$ implies $d(f(p_n), f(p_0)) < \varepsilon$. □

Problem 4.16. In Theorem 4.11 prove (b) implies (a). *Hint:* It is easier to prove the contrapositive: ∼(a) implies ∼(b). That is we assume that (a) is false and prove that (b) is false. If (a) is false there is $\varepsilon > 0$ such that for all $\delta > 0$ there is a $p \in E$ with $d(p, p_0)\delta$ and $d(f(p), f(p_0)) \geq \varepsilon$. Letting $\delta = 1/n$ shows there is a point p_n with $d(p_n, p_0) < 1/n$ and $d(f(p_n), f(p_0)) \geq \varepsilon$. Show that $\lim_{n\to\infty} p_n = p_0$, but $\lim_{n\to\infty} f(p_n) \neq f(p_0)$. □

Problem 4.17. In Theorem 4.11 prove (a) implies (c). *Hint:* Assume that f is continuous at p_0 and let $r > 0$. Letting $\varepsilon = r$ in the definition of f being continuous at p_0 gives a $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d(f(p), f(p_0)) < r$. If $p \in B(p_0, \delta)$, then $d(p, p_0) < \delta$ and therefore $d(f(p), f(p_0)) < r$. That is $p \in B(p_0, \delta)$ implies $f(p) \in B(f(p_0), r)$. Now use the definition of the preimage $f^{-1}\big[B(f(p_0), r)\big]$ to show this implies $B(p_0, \delta) \subseteq f^{-1}\big[B(f(p_0), r)\big]$. □

Problem 4.18. In Theorem 4.11 prove (c) implies (a). *Hint:* Assume that (c) holds and let $\varepsilon > 0$. Then in the statement of (c) let $r = \varepsilon$. As (c) holds, then is $\delta > 0$ such that $B(p_0, \delta) \subseteq f^{-1}\big[B(f(p_0), \varepsilon)\big]$. Let $p \in E$ with $d(p_0, p) < \delta$, then $p \in B(p, \delta) \subseteq f^{-1}\big[B(f(p_0), \varepsilon)\big]$. Then, by the definition of the $f^{-1}\big[B(f(p_0), \varepsilon)\big]$, this implies $f(p) \in B(f(p_0), \varepsilon)$. And you should be able to finish from here. $\square$

Proof of Theorem 4.11. Combining Problems 4.15, 4.16, 4.17, and 4.18 we have

$$(b) \iff (a) \iff (c)$$

which shows the three conditions are equivalent. $\square$

4.5. **Conditions equivalent to a function being continuous at all points.**

Definition 4.12. A map $f \colon E \to E'$ is ***continuous*** if and only if it is continuous at every point of E. $\square$

Theorem 4.13. *Let $f \colon E \to E'$ be a map between metric spaces. Then the following are equivalent.*

(a) f is continuous.
(b) f does the right thing to convergent sequences in E. That is if $\langle p_n \rangle_{n=1}^{\infty}$ is a convergent sequence in E, then

$$\lim_{n\to\infty} f(p_n) = f\big(\lim_{n\to\infty} p_n\big).$$

(c) If U is an open set in E', then the preimage $f^{-1}[U]$ is an open set in E. (That is preimages of open sets are open.)
(d) If F is a closed set in E', then the preimage $f^{-1}[F]$ is a closed set in E. (That is the preimages of closed sets are closed.)

Problem 4.19. Prove this. *Hint:* Note that Theorem 4.11 easily implies

$$(a) \iff (b) \iff (c)$$

and you can assume these equivalences. So all you have to do is prove (c) $\iff$ (d). $\square$

Now let us do some practice in using these equivalences.

Proposition 4.14. *Let $f \colon E \to E'$ and $g \colon E' \to E''$ be continuous maps between metric spaces. Then the composition $g \circ f \colon E \to E''$ is continuous.*

Problem 4.20. This is a special case of Theorem 4.8. In Problem 4.14 you gave a ε-δ proof. Now give a proof using that continuity is equivalent to doing the right thing by convergent sequences. *Hint:* It is enough to show that if f and g are continuous, and $\lim_{n\to\infty} p_n = p_0$ in E that $\lim_{n\to\infty} g(f(p_n)) = g(f(p_0))$ in E''. $\square$

Lemma 4.15. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be functions. Show that for and $S \subseteq E''$ that*

$$(g \circ f)^{-1}[S] = f^{-1}\big[g^{-1}[S]\big].$$

$\square$

Problem 4.21. Give anther proof of Proposition 4.14 using that continuity is equivalent to the preimages of open sets being open. *Hint:* Let f and g be continuous and $U \subseteq E''$ be open. All you need to show is that $(g \circ f)^{-1}[U]$ is open and the last lemma should make this easy.

4.6. **Continuous images of connected sets and the intermediate value theorem.**

Theorem 4.16. *Let E be a connected metric space and $f\colon E \to E'$ a continuous function. Then the image $f[E]$ is a connected subset of E'.*

Problem 4.22. Prove this. *Hint:* Towards a contradiction assume that $f[E]$ is not a connected. Then $f[E]$ has a disconnection, $f[E] = A \cup B$ (thus, by the definition of disconnection, A and B are nonempty, open in $f[E]$, and $A \cap B$). As f is continuous and the preimage of open sets by continuous functions is open the sets $f^{-1}[A]$ and $f^{-1}[B]$ are open in E. Use this to show $E = f^{-1}[A] \cup f^{-1}[B]$ is a disconnection of E, contradicting that E is connected. $\square$

Theorem 4.17 (General Intermediate Value Theorem)**.** *Let E be a connected metric space and let $f\colon E \to \mathbb{R}$ be continuous. Let $p_1, p_2 \in E$ with $f(p_1) < f(p_2)$. Then for every c with $f(p_1) < c < f(p_2)$ there is a at least one $p \in E$ with $f(p) = c$.*

Problem 4.23. Prove this. *Hint:* One way to do this is to use Theorem 4.16 to conclude that the image $f[E]$ is connected and then use Theorem 3.82 finish the proof. But it is possible to do a short proof that uses less machinery. Towards a contradiction assume that there is a c with $f(p_1) < c < f(p_2)$ and that there is no $p \in E$ with $f(p) = c$. Then $(-\infty, c)$ and (c, ∞) are open sets and, as preimages of open sets by continuous functions are open the sets $f^{-1}[(-\infty, c)]$ and $f^{-1}[(c, \infty)]$ are open in E. Show that

$$E = f^{-1}[(-\infty, c)] \cup f^{-1}[(c, \infty)]$$

is then a disconnection of E, contradicting that E is connected. $\square$

It is worth comparing this proof with the proof we gave earlier for Theorem 2.55 (the Lipschitz Intermediate Value Theorem). The theorem just proven is more general and now that we have the machinery of connected sets and that preimages of open sets by continuous functions are open the proof is much more transparent.

Definition 4.18. Let $\alpha, \beta \in \mathbb{R}$. Then $x \in \mathbb{R}$ is ***between*** α and β if and only if $\alpha < x < \beta$ or $\beta < x < \alpha$. $\square$

Theorem 4.19 (Intermediate Value Theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous with $f(a) \neq f(b)$. Then f takes on every value between $f(a)$ and $f(b)$. (That is if c is between $f(a)$ and $f(b)$ there is a $\xi \in (a,b)$ with $f(\xi) = c$.*

Proof. As the interval $[a,b]$ is connected, this is a special case of Theorem 4.17. □

The intermediate value theorem is useful in showing that equations have solutions, even in cases where we can not solve them explicitly. Here is an example: the equation $x^7 - 3x + 1 = 0$ has at least some solution with $0 < x < 1$. To see this note that $f(x) = x^7 - 3x + 1$ is continuous on $[0,1]$. Also $f(0) = 1$ is positive, and $f(1) = -1$ is negative. Therefore by Theorem 4.19 f takes on the value 0 at some point in $(0,1)$. That is there there is x_0 with $0 < x_0 < 1$ with $f(x_0) = x_0^7 - 3x_0 + 1 = 0$.

Problem 4.24. Show that the following have solutions.

(a) $x^3 = \sqrt{7+x}$ on the interval $[0,2]$. *Hint:* This can be rewritten as $x^3 - \sqrt{1+x} = 0$.
(b) $x^3 + 2x + 2 = 0$ on $[-2,2]$.
(c) $x^5 - 4x^3 + x - 9 = 0$ on $[-3,3]$.

Proposition 4.20. *Every polynomial of degree* 3 *has at least one real root. That is if $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with $a_3 \neq 0$ there is a real number x_0 with $f(x_0) = 0$.*

Proof. We wish to solve

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0.$$

As $a_3 \neq 0$ we can divide by a_3 and get the equivalent equation

$$x^3 + b_2x^2 + b_1x + b_0 = 0 \quad \text{where} \quad b_i = \frac{a_j}{a_3} \quad \text{for } j = 0,1,2.$$

Let

$$f(x) = x^3 + b_2x^2 + b_1x + b_0.$$

We will now find a c such that $f(c) > 0$ and $f(-c) < 0$ and therefore $f(x) = 0$ will have a solution $x = x_0$ with $-c < x_0 < c$ by the Intermediate value Theorem. We start by writing $f(x)$ as

$$f(x) = x^3\Big(1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3}\Big) = x^3q(x)$$

where

$$q(x) = 1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3}.$$

Now note if $|x| \geq 1$ that

$$\begin{aligned} q(x) &= 1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3} \\ &\geq 1 - \left|\frac{b_2}{x}\right| - \left|\frac{b_1}{x^2}\right| - \left|\frac{b_2}{x^3}\right| \\ &\geq 1 - \frac{|b_2|}{|x|} - \frac{|b_1|}{|x|} - \frac{|b_0|}{|x|} \qquad \text{(as } |x| \geq 1\text{)} \\ &= 1 - \frac{|b_2| + |b_1| + |b_0|}{|x|} \end{aligned}$$

Therefore if $|x| \geq 2(|b_2| + |b_1| + |b_0|)$ we have

$$q(x) \geq 1 - \frac{|b_2| + |b_1| + |b_0|}{|x|} \geq 1 - \frac{|b_2| + |b_1| + |b_0|}{2(|b_2| + |b_1| + |b_0|)} = \frac{1}{2}.$$

Whence if we set $c = 2(|b_2| + |b_1| + |b_0|)$ we have that

$$|x| \geq c \quad \text{implies} \quad q(x) > \frac{1}{2} > 0$$

Thus $q(c)$ and $q(-c)$ are both positive numbers and so

$$f(c) = c^3 q(c) > 0, \quad \text{and} \quad f(-c) = (-c)^3 q(-c) = -c^3 q(-c) < 0.$$

Therefore $f(x)$ change sign on $[-c, c]$ and f is continuous so by the Intermediate Value Theorem $f(x) = 0$ has a solution on $[-c, c]$. □

Problem 4.25. For any even integer $n = 2k$ given an example of a polynomial $f(x)$ such that $f(x) = 0$ has no solutions for any $x \in \mathbb{R}$. *Hint:* For $n = 2$ and example is $f(x) = x^2 + 1$. □

Theorem 4.21. *Let $f(x)$ be a polynomial of odd degree. Then there is a real number x_0 with $f(x_0) = 0$. That is all polynomial of odd degree have at least one root.*

Problem 4.26. Prove this for polynomial of degree 5. *Hint:* Look at the proof of Proposition 4.20. □

Proposition 4.22 (One dimensional fixed point theorem)**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $f\colon [a, b] \to [a, b]$ a continuous map. Then f has a fixed point in $[a, b]$. Then is there is a $\xi \in [a, b]$ with $f(\xi) = \xi$.*

Problem 4.27. Prove this. *Hint:* Apply the Intermediate Value Theorem to the function $g(x) = f(x) - x$. □

4.7. **Using continuous functions to show that sets are connected.** From Theorem 4.16 that the continuous image of a connected set is connected. We can combine this with Proposition 3.79 (about the union of mutually intersecting connected sets being connected) to show that many other sets that we feel should be connected are in fact connected. To start we construct some continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$.

Proposition 4.23. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ *and define* $f\colon \mathbb{R} \to \mathbb{R}^n$ *by*

$$f(t) = (1-t)\mathbf{a} + t\mathbf{b}.$$

Then f *is continuous.*

Problem 4.28. Prove this. *Hint:* Let $s, t \in \mathbb{R}$ and show

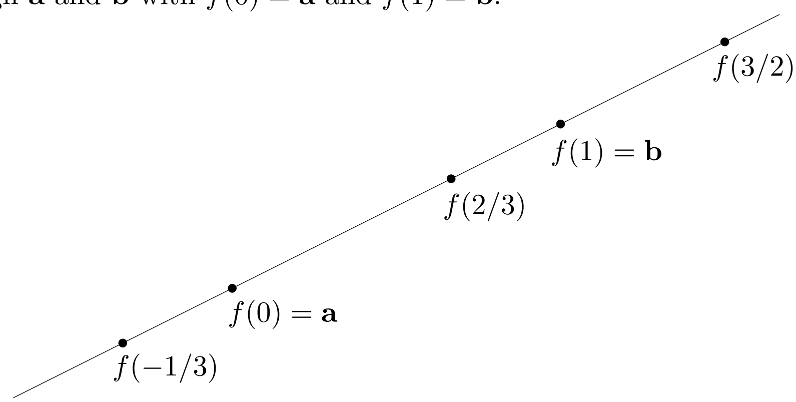
$$f(s) - f(t) = (s-t)(\mathbf{b} - \mathbf{a}).$$

Use this to show

$$|f(s) - f(t)| = \|\mathbf{b} - \mathbf{a}\| \, |s - t|$$

and therefore that f is Lipschitz and thus continuous. □

The function f has the geometric interpretation of parametrizing the line through $\mathbf{a}$ and $\mathbf{b}$ with $f(0) = \mathbf{a}$ and $f(1) = \mathbf{b}$.



Definition 4.24. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then the set

$$[\mathbf{a}, \mathbf{b}] = \{(1-t)\mathbf{a} + t\mathbf{b} : 0 \le t \le 1\}$$

is the ***segment*** from with endpoints $\mathbf{a}$ and $\mathbf{b}$. □

Problem 4.29. Show that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ that

$$[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}].$$

□

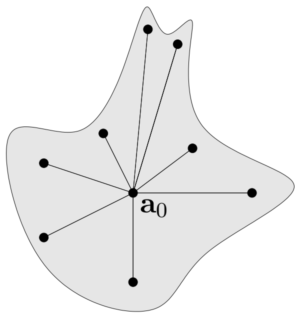
Proposition 4.25. *If* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$*, then the segment* $[\mathbf{a}, \mathbf{b}]$ *is a continuous image of the unit interval* $[0, 1]$ *and therefore is a connected set.*

Problem 4.30. Prove this. □

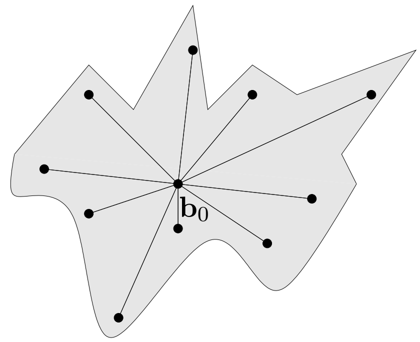
Definition 4.26. Let $S \subseteq \mathbb{R}^n$ and $\mathbf{a}_0 \in S$. Then S is ***starlike with respect to*** $\mathbf{a}_0$ if and only if for all $\mathbf{b} \in S$ we have

$$[\mathbf{a}_0, \mathbf{b}] \subseteq S.$$

That is for every point $\mathbf{b}$ in S, the set S contains the segment connecting $\mathbf{b}$ to $\mathbf{a}_0$. □



Starlike with respect to $\mathbf{a}_0$



Starlike with respect to $\mathbf{b}_0$

We call $S \subseteq \mathbb{R}^n$ ***starlike*** if and only if it is starlike with respect to some point.

Proposition 4.27. *Any nonempty starlike subset of* $\mathbb{R}^n$ *is connected.*

Problem 4.31. Prove this. *Hint:* Let $S \subseteq \mathbb{R}^n$ be startlike with respect to $\mathbf{a}_0 \in S$. Prove that

$$S = \bigcup_{\mathbf{b} \in S} [\mathbf{a}_0, \mathbf{b}].$$

Now use this and together with Proposition 4.25 and Proposition 3.79 to complete the proof.

4.8. **The continuous image of compact sets are compact and the existence of extreme values.**

Theorem 4.28 (Continuous images of compact sets are compact)**.** *Let* $f \colon E \to E'$ *be a continuous function between metric spaces and* $K \subseteq E$ *compact. Then the image* $f[K]$ *is a compact subset of* E'*.*

Problem 4.32. Give a proof of this using that the preimages of open sets by continuous functions are open. *Hint:* Let $K \subseteq E$ be compact and let $\mathcal{U}$ be an open cover of $f[K]$. We need to show that $\mathcal{U}$ has a finite subset that covers $f[K]$. Let

$$\mathcal{U}^* = \{f^{-1}[U] : U \in \mathcal{U}\}.$$

Then by Part (c) of Theorem 4.13 each $f^{-1}[U]$ is open. If $x \in K$, then $f(x) \in f[K]$ and $\mathcal{U}$ covers $f[K]$ and so $f(x) \in U$ for some $U \in \mathcal{U}$. Use this to show that $\mathcal{U}^*$ is an open cover of K. As K is compact there is a finite set $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that

$$K \subseteq f^{-1}[U_1] \cup f^{-1}[U_2] \cup \cdots \cup f^{-1}[U_n].$$

Use this to show

$$f[K] \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$$

and thus $\{U_1, U_2, \ldots, U_n\}$ is a finite subset of $\mathcal{U}$ that covers $f[K]$. Therefore every open cover of $f[K]$ has a finite subcover and so $f[K]$ is compact. $\square$

We have seen (combine Theorems 3.71 and 3.72) that a subset of a metric space is compact if and only if it is sequentially compact. So we could also prove Theorem 4.28 by showing that the continuous of a sequentially compact set is sequentially compact.

Problem 4.33. Prove that the continuous image of a sequentially compact set is sequentially is sequentially compact (which gives anther proof of Theorem 4.28). *Hint:* Let $f\colon E \to E'$ be a continuous map and $K \subseteq E$ a sequentially compact set. Let $\langle y_n \rangle_{n=1}^{\infty}$ be a subsequence in $f[K]$. We need to show that this sequence has subsequence that converges to a point of $f[K]$. As $y_n \in f[K]$ there is a $x_n \in K$ such that $y_n = f(x_n)$. Then $\langle x_n \rangle_{n=1}^{\infty}$ is sequence in K and K is sequentially compact and therefore there is a subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in K$. Now use Part (b) of Theorem 4.13 (that continuous functions do the right thing to convergent sequences) to conclude $\lim_{k\to\infty} y_{n_k} = f(x) \in f[K]$ to complete the proof. □

The following is a review or earlier material.

Lemma 4.29. *Let C be a compact subset of $\mathbb{R}$. Then C has a maximum and minimum element.*

Proof. We have seen that a compact set is sequentially compact (Theorem 3.72) and that sequentially compact subsets of a metric space are closed and bounded (Proposition 3.66). Therefore C is bounded and thus

$$\beta = \sup(C)$$

exists. Also if $r > 0$ and we had that $B(\beta, r) = (\beta - r, \beta + r)$ was disjoint from C, then $\beta - r$ would an upper bound for C that is less than β, contradicting that β is the least upper bound of C. Thus for all $r > 0$ we have $B(\beta, r) \cap C \neq \varnothing$. Thus β is an adherent point of C. But C is closed and thus contains all its adherent points. Therefore $\beta \in C$. But then $\beta \in C$ and for all $x \in C$ we have $x \leq \beta$. Thus β is the maximum of C.

A similar argument shows that $\alpha = \inf(C)$ is the minimum of C. □

The following generally considered one of the more important existence theorems in analysis.

Theorem 4.30. *Let E be a metric space, $f\colon E \to \mathbb{R}$ a continuous function. Let $K \subseteq E$ be compact. Then there are $a, b \in K$ such that*

$$f(a) \leq f(x) \leq f(b)$$

for all $x \in K$. (That is a continuous function on a compact set achieves its maximum and minimum.)

Problem 4.34. Prove this. *Hint:* Use one of our theorems to conclude that $f[K]$ is a compact subset of $\mathbb{R}$. Then by Lemma 4.29 a maximum, β, and a minimum, α. Thus $\alpha, \beta \in f[K]$. So there are $a, b \in K$ with $f(a) = \alpha$ and $f(b) = \beta$. □

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