Mathematics 555 Test #1: Solutions.

1. Let $f\colon (a,b) \to \mathbb{R}$ be a function.

(a) Define what it means for f to be ***continuous*** at x_0.

Solution. For all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon.$$

□

(b) Define what it means for f to be ***differentiable*** at x_0.

Solution. The function f is differentiable at $x - 0$ if and only if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. (The value of this limit is $f'(x_0)$, the derivative of f at x_0.) □

(c) Prove that if f is differentiable at x_0, then f is continuous at x_0.

Solution 1. We first prove the continuity of f at x_0 directly from the definition of the derivative. If f is differentiable at x_0 then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Therefore there is a $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \quad \text{implies} \quad \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < 1.$$

Thus if $0 < |x - x_0| < \delta_1$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| < |f'(x_0)| + 1.$$

Let $\varepsilon > 0$ and set

$$\delta = \min\left\{ \frac{\varepsilon}{|f(x_0)| + 1}, \delta_1 \right\}$$

Then if $0 < |x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| = |x - x_0| \left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \delta(|f'(x_0)| + 1) \le \varepsilon,$$

which shows f is continuous at x_0. □

Solution 2. As f is differentiable at x_0 we know that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists. Thus

$$\begin{aligned} \lim_{x \to x_0} f(x) &= \lim_{x \to x_0} \left(f(x_0) + (x - x_0) \frac{f(x) - f(x_0)}{x - x_0} \right) \\ &= f(x_0) + (0) f'(x_0) \\ &= f(x_0) \end{aligned}$$

and therefore f does the right thing to limits at x_0 which is equivalent to f being continuous at x_0. □

2. Let $f, g \colon I \to \mathbb{R}$ be functions on the open interval I. Assume that both f and g are differentiable at $a \in I$. Give an ε, δ proof that the product $p = fg$ is differentiable at a.

Solution. We start by using the adding and subtracting many times to get

$$\begin{aligned} &\frac{p(x) - p(x_0)}{x - x_0} - (f'(x_0)g(x_0) + f(x_0)g'(x_0)) \\ &= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} - (f'(x_0)g(x_0) + f(x_0)g'(x_0)) \\ &= \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) g(x_0) + \left(\frac{f(x) - f(x_0)}{x - x_0} \right) (g(x) - g(x_0)) \\ &\qquad + f(x_0) \left(\frac{g(x) - g(x_0)}{x - x_0} - g'(x_0) \right) \end{aligned}$$

By the triangle inequality this implies

$$\left|\frac{p(x)-p(x_0)}{x-x_0} - (f'(x_0)g(x_0)+f(x_0)g'(x_0))\right| \le E_1+E_2+E_2$$

where

$$\begin{aligned} E_1 &= \left|\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right) g(x_0)\right| \\ E_2 &= \left|\left(\frac{f(x)-f(x_0)}{x-x_0}\right)(g(x)-g(x_0))\right| \\ E_3 &= \left|f(x_0)\left(\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0)\right)\right| \end{aligned}$$

Let $\varepsilon > 0$. As $f'(x_0)$ exists there is a $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \quad \Longrightarrow \quad \left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \min\left\{1, \frac{\varepsilon}{3(|g(x_0)|+1)}\right\}$$

If this holds then

$$E_1 \le \frac{\varepsilon}{3(|g(x_0)|+1)} \le \frac{\varepsilon}{3}$$

Note also that $0 < |x-x_0| < \delta_1$ implies

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| < |f(x_0)|+1 \tag{1}$$

As g is continuous at x_0 there is a $\delta_1 > 0$ such that

$$|x-x_0| < \delta_2 \quad \Longrightarrow \quad |g(x)-g(x_0)| < \frac{\varepsilon}{3(|f(x_0)|+1)}$$

Combining this with the inequality (1) gives that if $0 < |x-x_0| < \min\{\delta_1, \delta_2\}$ that

$$E_2 < (|f(x_0)|+1)\left(\frac{\varepsilon}{3(|f(x_0)|+1)}\right) < \frac{\varepsilon}{3}.$$

Finally, as $g'(x_0)$ exists, there is a $\delta_3 > 0$ such that

$$0 < |x-x_0| < \delta_3 \quad \Longrightarrow \quad \left|\frac{g(x)-g(x_0)}{x-x_0} - g'(x_0)\right| < \frac{\varepsilon}{3(|fx_0|+1)}.$$

And therefore if $0 < |x - x_0| < \delta_3$

$$E_3 < |f(x_0)| \left(\frac{\varepsilon}{3(|fx_0| + 1)} \right) < \frac{\varepsilon}{3}.$$

Putting this all together, if $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ then

$$\left| \frac{p(x) - p(x_0)}{x - x_0} - (f'(x_0)g(x_0) + f(x_0)g'(x_0))i \right| < E_1 + E_2 + E_3 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore

$$p'(x_0) = \lim_{x \to x_0} \frac{p(x) - p(x_0)}{x - x_0} = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

as required. □

3. (a) State Taylor's theorem with Lagrange's form of the remainder.

Solution. Let I be an open interval and $f \colon I \to \mathbb{R}$ be a function which is $n+1$ times differentiable on I and $a \in I$. Then for any $x \in I$ there is a ξ between a and x such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_n(x)$$

where the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1}.$$

□

(b) Let $f \colon (0, 2)$ be a function with

$$f(1) = 1, \quad f'(1) = 1, \quad f''(1) = 2, \quad f'''(1) = 6.$$

What is the degree 3 Taylor polynomial of f at $x = 1$? (You do not have to simplify your answer.)

Solution. It is

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 + 1(.1) + 1(.1)^2 + 1(.1)^3 \\ &= 1.111 \end{aligned}$$

□

(c) Assume that also $|f^{(4)}(x)| \leq 1$ on $(0, 2)$ and show

$$|f(1.1) - 1.111| \leq \frac{1}{240{,}000}.$$

Solution. We use Lagrange's form of the remainder: There is a ξ between 1 and 1.1 such that

$$\begin{aligned} |f(1.1) - 1.111| &= |f(1.1) - T_3(1.1)| \\ &= \frac{|f^{(4)}(\xi)|}{4!}(.1)^4 \\ &\leq \frac{1}{24}(.1)^4 \\ &= \frac{1}{240{,}000}. \end{aligned}$$

□

4. (a) State the mean value theorem.

Solution. Let $f\colon [a, b] \to \mathbb{R}$ be differentiable on (a, b), continuous on $[a, b]$. Then there is a $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

□

(b) Show that for $f(x) = \sqrt[3]{x}$ that if $a, b \geq 8$, then

$$|f(b) - f(a)| \leq \frac{|b - a|}{12}.$$

Solution. The derivative of $f(x) = x^{\frac{1}{3}}$ is

$$f'(x) = \frac{1}{3}x^{\frac{-2}{3}} = \frac{1}{3x^{\frac{2}{3}}}.$$

This is a deceasing positive function on $[8, \infty)$ and thus for $\xi > 8$

$$0 < f'(\xi) \le \frac{1}{3(8)^{\frac{2}{3}}} = \frac{1}{12}.$$

Thus if $a, b \ge 8$ by the mean value theorem there is a ξ between a and b such that

$$|f(b) - f(a)| = |f'(\xi)||b - a| \le \frac{|b - a|}{12}.$$

$\square$

5. Let f be twice differentiable on $(-1, 3)$ with $f(1) = 1$ and $f'(1) = 3$, compute

$$\lim_{x \to 1} \frac{f(x) + 2 - 3x}{(x - 1)^2}.$$

Solution. We use L'Hôpital's rule.

$$\begin{aligned} \lim_{x \to 1} \frac{f(x) + 2 - 3x}{(x - 1)^2} &= \lim_{x \to 1} \frac{f'(x) - 3}{2(x - 1)} && \text{(0/0 limit so L'Hôpital applies)} \\ &= \lim_{x \to 1} \frac{f''(x)}{2} && \text{(L'Hôpital again)} \\ &= \frac{f''(1)}{2}. \end{aligned}$$

$\square$

6. Let $f\colon (a, b) \to \mathbb{R}$ be a differentiable function with $f' \ne 0$ in (a, b) and with $f'(x) = 3 + f(x)^3$. Let g the inverse of f, that is $g(f(x)) = x$.

(a) How do we know g is differentiable?

Solution. We have a theorem that tells us if f in differentiable and has an inverse, g, then g is differentiable at all points $f(x)$ where $f'(x) \ne 0$ and $g'(f(x)) = 1/f(x)$. $\square$

(b) Find the derivative of g.

Solution 1. Take the derivative of $g(f(x)) = x$ to get

$$g'(f(x))f'(x) = x.$$

Use the differential equation for f to rewrite this as

$$g'(f(x))(3 + f(x)^3) = 1$$

so that

$$g'(f(x)) = \frac{1}{3 + f(x)^3}.$$

The change of variable $y = f(x)$ then gives

$$g'(y) = \frac{1}{3 + y^3}.$$

□

Solution 2. Since g is the inverse of g we have also have $f(g(x)) =$. Take the derivative of this to get

$$f'(g(x))g'(x) = 1.$$

The differential equation for $f(x)$ implies

$$f'(g(x)) = 3 + f(g(x))^3 = 3 + x^2.$$

Combining these gives

$$(3 + x^3)g'(x) = 1$$

and therefore

$$g'(x) = \frac{1}{3 + x^3}$$

□

7. (a) State the Cauchy mean value theorem.

Solution. Let $f, g\colon [a, b] \to \mathbb{R}$ be differentiable on (a, b) and continuous on $[a, b]$. Then there is a $\xi \in (a, b)$ so that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

It was also acceptable to write this as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

□

(b) Let f, g be differentiable functions on an open interval I that $|f'(x)| \leq 2|g'(x)|$ and $g'(x) \neq 0$ for all $x \in I$. Show that for all $a, b \in I$

$$|f(b) - f(a)| \leq 2|g(b) - g(a)|.$$

Solution. As $g'(x) \neq 0$ on $(a.b)$ the mean value theorem tells use that $g(a) \neq g(b)$. Therefore we can divide by $g(b) - g(a)$. Using the Cauchy mean value theorem and $|f'(x)| \leq 2|g'(x)|$

$$\left|\frac{f(b) - f(a)}{g(b) - g(a)}\right| = \left|\frac{f'(\xi)}{g'(\xi)}\right| \leq \frac{2|g'(\xi)|}{|g'(\xi)|} = 2$$

Now multiplication by $|g(b) - g(a)|$ gives the result. $\square$