Mathematics 550 Homework.

We have derived both the two and three dimensional versions of the Divergence Theorem.

Theorem 1 (Two dimensional Divergence Theorem)**.** *Let D be a reasonable domain in $\mathbb{R}^2$ and let $\mathbf{N}$ be the outward unit normal the boundary curve ∂D. Then for any continuously differentiable vector field $\mathbf{F}$*

$$\oint_{\partial D} \partial F \cdot \mathbf{n}\, ds = \iint_D \operatorname{div}(\mathbf{F})\, dA$$

where ds is integration with respect to arclength along ∂D. □

Where the divergence is defined by

$$\operatorname{div}(P\mathbf{i} + Q\mathbf{j}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Theorem 2 (Three dimensional Divergence Theorem)**.** *Let D be a reasonable domain in $\mathbb{R}^n$ and let $\mathbf{n}$ be the outward unit normal along the boundary surface ∂D. Then for any continuously differentiable vector field $\mathbf{F}$*

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where $d\sigma$ is integration with respect to surface area on ∂D. □

This time the divergence is

$$\operatorname{div}(P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Problem 1. Let B be the ball in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 \le r^2$. That is the ball of radius r centered at the origin. Then ∂B is the sphere of radius r at the origin. Let $\mathbf{V}$ be the vector field

$$\mathbf{V} = \langle x, y, z\rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

(a) Show, or at least draw a picture that show, that on ∂B

$$\mathbf{n} = \frac{1}{r}\mathbf{V}.$$

(b) Compute $\operatorname{div}(\mathbf{V})$.

(c) Use the Divergence Theorem to show

$$\text{Volume}(B) = \frac{r}{3}\,\text{Area}(\partial B).$$

This is due to Archimedes and is equivalent to our usual formula for the volume of a sphere.

Problem 2. The problem refers to Figure 1. Let D be the region as shown in the figure. Then the arrows show the direction we are moving along the boundary curves and the direction of the unit normals. Let $\mathbf{V}$ be a vector field with

$$\operatorname{div}(\mathbf{F}) = 0 \qquad \text{in } D.$$

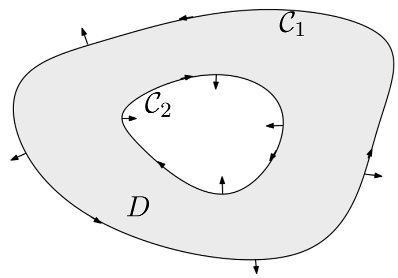

FIGURE 1. The shaded region is between two curves, $\mathcal{C}_1$, on the outside, and $\mathcal{C}_2$, on the inside. Our convention is that we move along boundary curves in the direction that keeps the inside of the region on the left. This means that the outward pointing normal is on the right.

(a) Use the Divergence Theorem to show

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{n}\, ds + \oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{n}\, ds = 0$$

(b) Let $\mathcal{C}_0$ be the curve $\mathcal{C}_2$ traversed in the opposite direction. Explain why

$$\oint_{\mathcal{C}_0} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{n}\, ds$$

Problem 3. Let

$$\mathbf{V} = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}.$$

Let $\mathcal{C}_r$ be the circle $x^2 + y^2 = r^2$. Compute

(a) Show

$$\oint_{\mathcal{C}_r} \mathbf{V} \cdot \mathbf{n}\, ds = 2\pi$$

(b) Show $\operatorname{div}(\mathbf{V}) = 0$.

(c) Let D_r be the disk defined by $x^2 + y^2 \le r^2$. Then $\partial D_r = \mathcal{C}_r$. As $\operatorname{div}(\mathbf{V}) = 0$ the Divergence Theorem should imply

$$\oint_{\mathcal{C}_r} \mathbf{V} \cdot \mathbf{n}\, ds = \iint_{D_r} \operatorname{div}(\mathbf{V})\, dA = 0,$$

which would contradict part (a) of this problem. Explain why the Divergence Theorem does not apply in this case.

Problem 4. Let $\mathbf{V}$ be the vector field of Problem 3. Let $\mathcal{C}$ be any simple closed curve around the origin is in Figure 2. Put together some of the facts from the previous problems to show

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds = 2pi.$$

This the value of the integral does not depend on the shape of the curve, only that it surrounds the origin. □

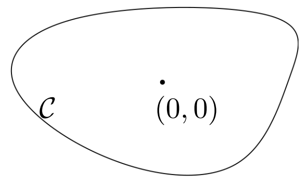

FIGURE 2. The simple closed curve surrounds the origin.

Problem 5. Assume that the temperature at a point of $\mathbb{R}^3$ is given by

$$T(x, y, z) = \frac{1}{1 + x^2 + 2y^2 + 3z^2}.$$

(a) Compute the gradient ∇T is T.
(b) If a heat loving bug is at the point $(1, -1, 5)$ and it moves in the direction of the greatest increase of the temperature, then what direction does it move?

Problem 6. Let f and g be functions with continuous partial derivatives. Prove the following form of the product rule for gradients:

$$\nabla(fg) = f\nabla g + g\nabla f.$$

Problem 7. Let f be a differentiable vector field and $\mathbf{V}$ a differentiable vector field defined on a reasonable domain D.
(a) Prove the following product rule:

$$\operatorname{div}(f\mathbf{V}) = \nabla f \cdot \mathbf{V} + f \operatorname{div}(\mathbf{V}).$$

(b) Use this to prove the following version of integration by parts:

$$\iiint_D f \operatorname{div}(\mathbf{V})\, dV = \iint_{\partial D} f\mathbf{V} \cdot \mathbf{n}\, dA - \iiint_D \nabla f \cdot \mathbf{V}\, dV.$$

Problem 8. Let

$$\mathbf{V} = (2xy^3z^4 + y)\mathbf{i} + (3x^2y^2z^4 + x + 2z^2)\mathbf{j} + (4x^2y^3z^3 + 4yz)\mathbf{k}.$$

Show $\mathbf{V}$ is conservative and find its potential.