Mathematics 550 Homework.

Here is an example: Find the maximum and minimum of $f(x,y,z) = x^2 + (y-1)^2 + (z-4)^2$ on the sphere defined by $x^2 + y^2 + z^2 = 68$.

In this case our objective function is

$$f(x,y,z) = x^2 + (y-1)^2 + (z-4)^2$$

and the constraint is

$$g(x,y,z) = x^2 + y^2 + z^2 = 160$$

. The Lagrange multiplier equations are

$$\begin{aligned}
&(1) \qquad f_x = \lambda g_x \qquad \Longrightarrow \qquad 2x = 2\lambda x \\
&(2) \qquad f_y = \lambda g_y \qquad \Longrightarrow \qquad 2(y-1) = 2\lambda y \\
&(3) \qquad f_z = \lambda g_z \qquad \Longrightarrow \qquad 2(z-4) = 2\lambda z
\end{aligned}$$

Case 1: $x \neq 0$. If this holds we can cancel the x in equation (1) to get $\lambda = 1$. Using $\lambda = 1$ in equation (2) gives

$$2(y-1) = 2y$$

which leads to the contradiction $-2 = 0$. So we conclude $x = 0$, that is $x = 0$ on any critical point for this problem.

Case 2: $x = 0$. Then solving for λ in equations (2) and (3) gives

$$\lambda = \frac{y-1}{y} = \frac{z-4}{z}.$$

Multiply by yz to get

$$z(y-1) = y(z-4)$$

and therefore

$$z = 4y.$$

Using this and $x = 0$ in the constraint equation gives

$$0^2 + y^2 + (4y)^2 = 17y^2 = 68.$$

Thus $y^2 = 4$ and so $y = \pm 2$ and $z = 4y = \pm 8$. Thus we have the two critical points

$$(0,2,8) \quad \text{and} \quad (0,-2,-8).$$

At these values

$$f(0,2,8) = 17$$
$$f(0,-2,-8) = 153.$$

So the maximum of 153 which occurs at $(0,-2,-8)$ and the minimum is 17 which occurs at $(0,2,8)$.

Problem 1. Find the maximum and minimum of $f(x,y,x) = x^2 + (y-1)^2 + (z-2)^2$ on the ball $B = \{(x,y,z) : x^2 + y^2 + z^2 \leq 500\}$.*Hint:* You have to look for critical points both inside of B and on its boundary. □

Problem 2. Find the maximum of $f(x, y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$. □

Problem 3. Let $M > 0$ be a constant find the minimum of sum of three positive number whose product is M. (That is minimize $x + y + z$ subject to the constraint $xyz = M$. What are the numbers x, y, z that do the minimization. □

Problem 4. Let $f(x, y)$ be a function with continuous first partial derivatives and assume $\dfrac{\partial f}{\partial y} \neq 0$. Let $\phi(x)$ be a function such that

$$f(x, \phi(x)) = c \tag{4}$$

holds on some interval (a, b) where c is a constant.

(1) Take the derivative $\dfrac{d}{dx}$ of equation (4) and use the result to prove

$$\phi'(x) = -\frac{f_x(x, \phi(x))}{f_y(x, \phi(x))}.$$

(2) (You should consider this somewhat of a challenge problem). Take the second derivative of (4) to get a formula for $\phi''(x)$.