Key to take home part of Test 1.

This is due at the beginning of class on Monday. I am assuming that you can do the problems. So I what nice write up. This means you should do the problem on scratch paper first and then turn in a good copy. (Which I am happy to say many of you already do on the homework,)

Problem 1. A fact we have used many times is that partial derivatives commute. That is

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

Giving a fully rigorous proof of this was a long time in the making, the first incomplete proof seems to have been given by Euler in 1740 and the first proof up to modern standards was given by Schwarz in 1873. See the Wikipedia article:

`https://en.wikipedia.org/wiki/Symmetry_of_second_derivatives`

for the history. Here you will give a proof that would be good enough for Euler and is one that I find easy to follow. We know the official definitions of the partial derivative are

$$\frac{\partial f}{\partial x}(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y}(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

We will abbreviate these as

$$\frac{\partial f}{\partial x}(x,y)\approx\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

and

$$\frac{\partial f}{\partial y}(x,y)\approx\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

when $\Delta x,\Delta y\approx 0$.

Then we have that when $\Delta x,\Delta y\approx 0$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y)\approx\frac{\dfrac{\partial f}{\partial y}(x+\Delta x,y)-\dfrac{\partial f}{\partial y}(x,y)}{\Delta x}$$

Now

$$\frac{\partial f}{\partial y}(x+\Delta x,y)\approx\frac{f(x+\Delta x,y+\Delta y)-f(x+\Delta x,y)}{\Delta y}$$

and

$$\frac{\partial f}{\partial y}(x,y)\approx\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

Put all these facts together to get a formula

$$\begin{aligned}&\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y)\\&\approx \mathcal{F}(\Delta x, \Delta y, f(x,y), f(x+\Delta x, y), f(x, y+\Delta y), f(x+\Delta x, y+\Delta y))\end{aligned}$$

where $\mathcal{F}$ only depends on the indicated quantities.

Now do a similar calculation for

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y).$$

The two formulas should be the same. □

Solution. We first have

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) &\approx \frac{\frac{\partial f}{\partial y}(x+\Delta x, y) - \frac{\partial f}{\partial y}(x,y)}{\Delta x}\\ &\approx \frac{\frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)}{\Delta y} - \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}}{\Delta x}\\ &= \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x,y)}{\Delta x \Delta y}\end{aligned}$$

This is the formula $\mathcal{F}(\Delta x, \Delta y, f(x,y), f(x+\Delta x, y), f(x, y+\Delta y), f(x+\Delta x, y+\Delta y))$ were suppose to find. If you did not explicitly give this formula you lost five points. Likewise if you did not explicitly give the for the doing the partial derivatives in the other direction.

Now we do the calculation for the partial derivatives in the other order.

$$\begin{aligned}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) &\approx \frac{\frac{\partial f}{\partial x}(x, y+\Delta y) - \frac{\partial f}{\partial x}(x,y)}{\Delta y}\\ &\approx \frac{\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} - \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}}{\Delta y}\\ &= \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x,y)}{\Delta x \Delta y}\end{aligned}$$

So we end up with the same formula for $\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}$. Thus

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

as required. □

Problem 2. Let $\mathcal{C}_R$ be the circle $x^2+y^2 = R^2$ transversed counter clockwise. Compute the line integral

$$\oint_{\mathcal{C}_R} \frac{-y\,dx + x\,dy}{x^2+y^2}.$$

Solution. This circle is parameterized by

$$x(t) = R\cos(t), \quad y(t) = R\sin(t), \qquad 0 \le t \le 2\pi.$$

$\square$

Then

$$dx = -R\sin(t)\,dt, \qquad dy = R\cos(t)\,dt$$

Therefore

$$\begin{aligned}\oint_{\mathcal{C}_R} \frac{-y\,dx + x\,dy}{x^2+y^2} &= \int_0^{2\pi} \frac{-\Big(-(R\sin(t))(-R\sin(t)) + (R\cos(t))(R\cos(t))\Big)\,dt}{(R\cos(t))^2 + (R\sin(t))^2} \\ &= \int_0^{2\pi} \frac{R^2}{R^2}\,dt \\ &= 2\pi.\end{aligned}$$