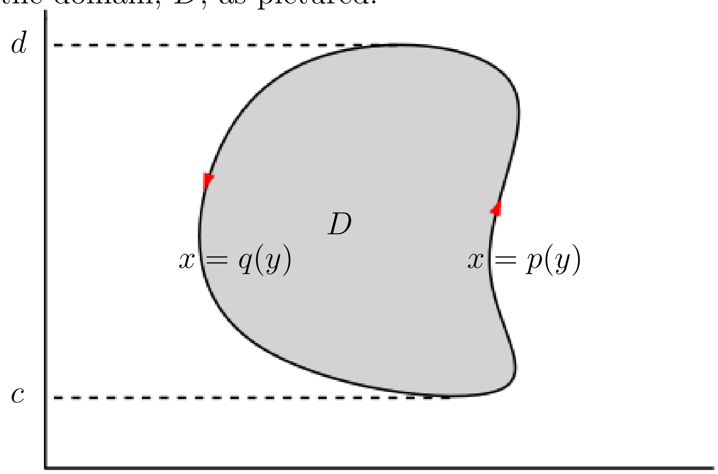
Answer key to in class part of test 1, Math 550

1. For the domain, D, as pictured:



let $R(x,y)$ be a differentiable function on D. Prove

$$\oint_{\partial D} R(x,y)\,dy = \iint_D \frac{\partial R}{\partial x}(x,y)\,dx\,dy$$

where ∂D is the boundary curve of D.

Solution. Our convention is that we orient the boundary so that the interior of D is to the left of the direction of motion. I have added arrows in red to the figure showing the orientation. Then the line integral splits into two pieces one parameterized by $\mathbf{r}_1(y) = (p(y), y)$, with $c \le y \le d$ and with y going from c to d. The other piece is parameterized by $\mathbf{r}_2(y) = (q(y), y)$ with $c \le y \le d$, by this time y is going from d to c. Then using the Fundamental Theorem of Calculus (FTC)

$$\begin{aligned}
\oint_{\partial D} R(x,y)\,dy &= \int_c^d R(p(y),y)\,dy + \int_d^c R(p(y),y)\,dy \\
&= \int_c^d R(p(y),y)\,dy - \int_c^d R(p(y),y)\,dy \\
&= \int_c^d \big(R(p(y),y) - R(q(y),y)\big)\,dy \\
&= \int_c^d \int_{q(y)}^{p(y)} \frac{\partial R}{\partial x}(x,y)\,dx\,dy \qquad \text{(by FTC)} \\
&= \iint_D \frac{\partial R}{\partial x}(x,y)\,dx\,dy.
\end{aligned}$$

□

2. Find a potential, fm for the vector field $\mathbf{V} = (2xy+2)\mathbf{i} + (x^2+3)\mathbf{j}$. That is find a function f such that $\nabla f = \mathbf{V}$.

Solution. A bit more explicitly we are looking for a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x}(x,y) = 2xy + 2 \qquad \frac{\partial f}{\partial y}(x,y) = x^2 + 3.$$

We then have

$$f(x,y) = \int (2xy+2)\,dx = x^2 + 2x + c(y)$$

where the constant of integration $c = c(y)$ depends on y. Then take $\dfrac{\partial}{\partial y}$ of this to get

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\big(x^2y + 2x + c(y)\big) = x^2 + c'(y) = x^2 + c'(y) = x^2 + 3$$

This gives $c'(y) = 3$ and therefore $c(y) = 3y + c_0$ where c_0 is a constant. Thus

$$f(x,y) = x^2y + 2x + 3y + c_0$$

is the general solution. As the problem only asked for a solution just giving $f = x^2y + 2x + 3y$ was fine. □

3. Compute the line integral

$$\int_{\mathcal{C}} xy\,dx + x\,dy$$

where $\mathcal{C}$ is the line segment from $(1,0)$ to $(2,3)$.

Solution. The line segment between these to points is parameterized by

$$\mathbf{r}(t) = (1-t)(1,0) + t(2,3) = (1+t, 3t).$$

That is

$$\begin{aligned} x &= t+1 & dx &= dt \\ y &= 3t & dy &= 3dt \end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathcal{C}} xy\,dx + x\,dy &= \int_0^1 (t+1)(3t)\,dt + (t+1)\,3dt \\ &= \int_0^1 (3t^2+6t+3)\,dt \\ &= (t^3+3t^2+3t)\Big|_{t=0}^{1} \\ &= 7\end{aligned}$$

□

4. (a) Give a parameterization of the ellipse:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Solution. Probably the easiest parameterization is

$$x(t) = 2\cos(t), \quad y(t) = \sin(t), \qquad 0 \le t \le 2\pi.$$

□

(b) Set up the integral for the arclength, L, of this ellipse. (Do not try to evaluate the integral.)

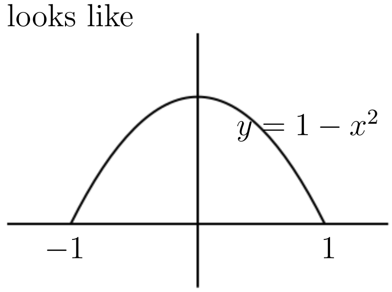
Solution. Using the usual formula for arclength

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\,dt$$

□

5. Compute the integral $\iint_D x^2\,dA$ where D region above the x-axis and below the parabola $y = 1 - x^2$.

Solution. The graph looks like



If we do dy first (that is the inner integral), then y goes from 0 to $1-x^2$. Then x goes from -1 to 1.

$$\begin{aligned}
\int_D x^2\,dA &= \int_{-1}^{1}\int_0^{1-x^2} x^2\,dy\,dx \\
&= \int_{-1}^{1} x^2 y\Big|_{y=0}^{1-x^2}\,dx \\
&= \int_{-1}^{1} x^2(1-x^2)\,dx \\
&= \int_{-1}^{1} (x^2-x^4)],dx \\
&= \left(\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_{x=-1}^{1} \\
&= \frac{4}{15}.
\end{aligned}$$

□

6. What is the chain rule for $\dfrac{d}{dt}f(x(t),y(t))$?

Solution. This can be written in several different ways:

$$\begin{aligned}
\frac{d}{dt}f(x(t),y(t)) &= \frac{\partial f}{\partial x}(x(t),y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t))\frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}(x(t),y(t))x'(x) + \frac{\partial f}{\partial y}(x(t),y(t))y'(t)
\end{aligned}$$

Or if you prefer the vector form set $\mathbf{r}(t) = (x(t),y(t))$, then the chain rule is

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t).$$

□

7. Use the chain rule and the Fundamental Theorem of Calculus to explain why if $\mathbf{r}\colon [a,b]\to\mathbf{R}^2$ is a parameterization of the curve $\mathcal{C}$ and $f\colon \mathbf{R}^2\to\mathbf{R}$ is a differentiable function then

$$\int_{\mathcal{C}} \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Solution. Use the definitions. To start ∇f is the vector fields

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right\rangle$$

Let $\mathbf{r}(t) = (x(t), y(t))$, then

$$d\mathbf{r}(t) = \langle x'(t), y'(t)\rangle\, dt.$$

Putting these together gives

$$\begin{aligned}
\nabla f \cdot \mathbf{r} &= \left\langle \frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) \right\rangle \cdot \langle x'(t), y'(x)) \rangle \; dt \\
&= \frac{\partial f}{\partial x}(x(t), y(t)) x'(t)\, dt + \frac{\partial f}{\partial y}(x(t), y(t)) y'(t)\, dt \\
&= \frac{d}{dt} f(x(t), y(t))\, dt \\
&= \frac{d}{dt} f(\mathbf{r}(t))\, dt.
\end{aligned}$$

where the last set is by the chain rule.

Then by the Fundamental Theorem of Calculus

$$\begin{aligned}
\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).
\end{aligned}$$

□

8. Set up the triple integral for

$$\iiint_D f(x, y, z)\, dx\, dy\, dz$$

over the region defined by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $3x + 2y + z \leq 6$.

Solution. We went over this problem in class. People wrote the correct answer one of two ways:

$$\int_0^6 \int_0^{\frac{6-z}{2}} \int_0^6 f(x, y, z)\, dx\, dy,\ dz$$

□

or

$$\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{6-3x-3y} f(x, y, z)\, dz\, dy\, dx.$$