SCHUR TYPE COMPARISON THEOREMS FOR AFFINE CURVES

RALPH HOWARD

Abstract. Let $c, \overline{c}\colon [a,b] \to \mathbb{R}^2$ are two convex planar curve parameterized by affine arc length and A is the area bounded by the image of c and its endpoints and $\overline{A}$ the corresponding area for $\overline{c}$. If the affine curvatures are related by $\varkappa(s) \le \overline{\varkappa}(s)$, then $A \ge \overline{A}$. If either $\varkappa$ or $\overline{\varkappa}$ is C^1 then equality holds if and only if $\overline{c}$ is the image of c under a special affine motion. Also for any point of a convex curve we define *adapted affine coordinates* centered at the point and give sharp estimates on the coordinates of the curve in terms of bounds on the curvature. Proving these bounds involves generalizing classical comparison theorems of Strum-Liouville type to higher order and nonhomogenous equations. These estimates allow us to give sharp upper bounds on the areas of triangles with whose vertices are on a curve in terms of the curvature and a affine length between the points.

1. Introduction.

The results in this paper were motivated by wanting to give extensions, or more precisely analogues, of some of the results of the Euclidean differential geometry of curves to the affine setting. Let $\mathcal{C}$ be an embedded connected curve in the plane. Then $\mathcal{C}$ is ***convex*** if and only if $\mathcal{C}$ is on the boundary of its convex hull. The curve is a ***closed convex*** curve if and only if it is the boundary of a bounded convex set.

A basic result in the Euclidean differential geometry of curves is the comparison theorem of A. Schur: if $\mathcal{C}$ and $\overline{\mathcal{C}}$ are two convex non-closed curves of the same length and the curvature of $\overline{\mathcal{C}}$ is pointwise greater than the curvature of $\mathcal{C}$, then the distance between the endpoints of $\overline{\mathcal{C}}$ is smaller than the distance between the endpoints of $\mathcal{C}$. That is the greater the curvature the smaller the distance between the endpoints. (See [3, Page 36] or [6, Thm. 2-19, Page 31] for the precise statement

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and a proof.) The original proof is in [10]. Some extensions and generalization are given in [12] (higher dimensions and minimal regularity), [5] (version in hyperbolic space) and [9] (version in the Lorentzian plane).

In affine geometry the distance between points in the plane is not defined, but the area bounded by the segment joining the endpoints of the curve and the curve (see Figure 1) is defined. Recall a ***special affine motion*** of $\mathbb{R}^2$ is a map of the form $v \mapsto Mv + b$ where M is a linear map with $\det(M) = 1$ and $b \in \mathbb{R}^2$. (See Section 2 for our conventions and definitions of affine arc length and affine curvature.) The affine version of Schur's Theorem is the greater curvature the smaller the bounded area:

Theorem A. *Let $c, \overline{c}\colon [a,b] \to \mathbb{R}^2$ be convex curves parameterized by affine arc length with affine curvatures $\varkappa$ and $\overline{\varkappa}$ respectively and let A and $\overline{A}$ be the respective areas bounded by the curves and the segment between their endpoints as in Figure 1 (see Section 2 for a precise definition of A and $\overline{A}$). Assume that $\overline{\varkappa}$ satisfies any one of the following three conditions (a) $\overline{\varkappa}$ is constant, (b) $\overline{\varkappa} \le 0$, or (c) $\overline{\varkappa} \le k_1$ where k_1 is a positive constant with $k_1 \le (\pi/(b-a))^2$. Then*

$$\begin{aligned} \varkappa \le \overline{\varkappa} \text{ on } [a,b] \quad &\textit{implies} \quad A \ge \overline{A} \\ \varkappa \ge \overline{\varkappa} \text{ on } [a,b] \quad &\textit{implies} \quad A \le \overline{A} \end{aligned}$$

If $\overline{\varkappa}$ is C^1, then $A = \overline{A}$ implies c is the image of $\overline{c}$ by a special affine motion.

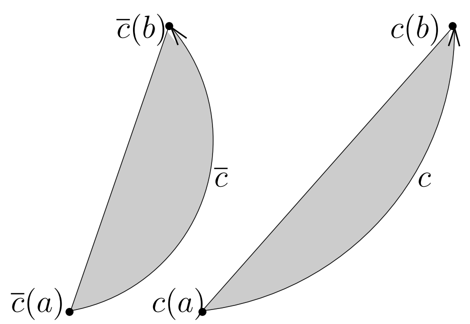

Figure 1. The curves c and $\overline{c}$ showing the areas bounded by the curves and the secant segments between their endpoints.

This is a special case of Theorem 4.1 below. We also give bounds on the coordinates of a curve in near a point in terms of coordinates adapted to curve at the point. To state a special case of this result, let

k be a real number and define functions $\overline{x}_k$ and $\overline{y}_k$ by

$$\overline{x}_k(s) = \begin{cases} \dfrac{\sin(\sqrt{k}\, s)}{\sqrt{k}}, & k > 0; \\ s, & k = 0; \\ \dfrac{\sinh(\sqrt{|k|}\, s)}{\sqrt{|k|}}, & k < 0. \end{cases} \tag{1.1}$$

$$\overline{y}_k(s) = \begin{cases} \dfrac{1-\cos(\sqrt{k}\, s)}{k}, & k > 0; \\ \dfrac{s^2}{2}, & k = 0; \\ \dfrac{1-\cosh(\sqrt{|k|}\, s)}{k}, & k < 0. \end{cases} \tag{1.2}$$

Let $\overline{c}_k \colon \mathbb{R} \to \mathbb{R}^2$ be the curve $\overline{c}_k(s) = (\overline{x}_k(s), \overline{y}_k(s))$. Then $\overline{c}_k$ is an affine unit speed curve with constant affine curvature k and

$$\overline{c}_k(0) = (0,0), \quad \overline{c}_k'(0,0) = (1,0), \quad \overline{c}''(0) = (0,1).$$

This parameterizes the connected component of the conic $x^2 + ky^2 - 2y = 0$ containing the origin. The following is a special case of Theorem 5.5.

Theorem B. *Let $c \colon (-L, L) \to \mathbb{R}^2$ be affine unit speed curve with*

$$c(0) = (0,0), \quad c'(0) = (1,0), \quad c''(0) = (0,1)$$

and whose curvature satisfies $k_0 \le \varkappa(s) \le k_1$ for constants k_0 and k_1 where $k_1 \le (\pi/2L)^2$. Then on the interval $(-L, L)$ the curve is a graph $y(s) = f(x(s))$ and the coordinates satisfy

$$\overline{x}_{k_1}(|s|) \le |x(s)| \le \overline{x}_{k_0}(|s|)$$
$$\overline{y}_{k_1}(s) \le y(s) \le \overline{y}_{k_0}(s).$$

(See Figure 2).

Theorem 5.5 also gives necessary and sufficient conditions for equality to hold.

These results are used to give estimates on areas of triangles with vertices on a curve in terms of the affine length between the vertices and bounds on the affine curvature (Proposition 4.3 and Theorem 5.8). In a future paper we plan to use these results to give bounds on the number of lattice points on a curve in terms of its affine length and affine curvature bounds.

The proofs are based on generalizations of Sturm type comparison theorems for homogeneous second order differential equations to equations that are non-homogeneous and of higher order. If I is an interval

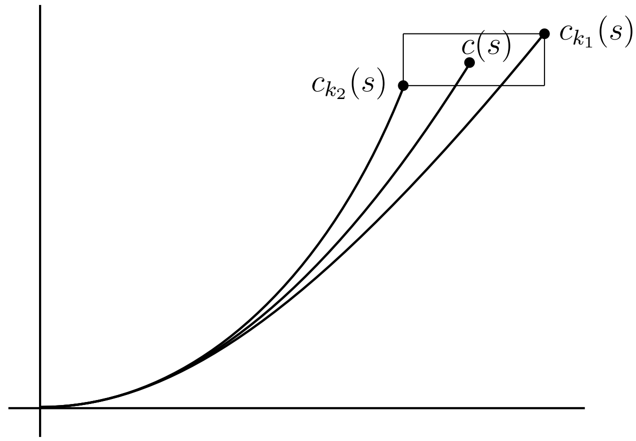

Figure 2. The curve c has is affine unit speed and its affine curvature satisfies $k_1 \le \varkappa \le k_2$, $c(0) = (0,0)$, $c'(0) = (1,0)$, and $c''(0) = (0,1)$. Assume $k_1 \le (\pi/2L)^2$ (which holds for all L when $k_1 \le 0$). Then when $|s| \le L$ the point $c(s)$ is in the rectangle with sides parallel to the axis and two of its vertices $c_{k_1}(s)$ and $c_{k_0}(s)$.

an $c\colon I \to \mathbb{R}^2$ is smooth curve with affine curvature $\varkappa$, then

$$c'''(s) + \varkappa(s)c'(s) = 0$$

and therefore the coordinate functions of c satisfy the third order scalar equation $u'''(s) + \varkappa(s)u'(s) = 0$. In Theorem 1 let $A = A(b)$ be viewed as a function of the endpoint $c(b)$, then a special case of Theorem 2.2, is that A satisfies the third order initial value problem

$$A''' + \varkappa A' = \frac{1}{2}, \qquad A(a) = A'(a) = A''(a) = 0.$$

Thus Theorems A and B follow from theorems that give pointwise comparisons of solutions to equations

$$\overline{y}''' + \overline{\varkappa}\,\overline{y}' = f, \quad y''' + \varkappa y' = f, \tag{1.3}$$

$$\overline{y}(0) = \overline{y}(0), \quad \overline{y}'(0) = y'(0) \quad \overline{y}''(0) = y''(0). \tag{1.4}$$

under the assumptions $\overline{\varkappa} \ge \varkappa$ or $\overline{\varkappa} \le \varkappa$. Section 3 has a general theory for comparisons of solutions to linear ordinary differential equations which covers theses cases and maybe of independent interest. The main idea in proving these comparisons is to rewrite the initial value problem $\overline{u}''' + \overline{\varkappa}\,\overline{u}' = f$, $\overline{u}(a) = u'(a) = \overline{u}''(a) = 0$ as an integral equation

$$\overline{u}(s) = \int_a^s \overline{K}(s;t) f(t)\, dt.$$

With this notation the equations (1.3) and (1.4) imply

$$\overline{y}(s) - y(s) = -\int_a^s \overline{K}(s;t)(\overline{\varkappa}(t) - \varkappa(t))y'(t)\,dt. \tag{1.5}$$

The main steps involved are finding conditions on $\overline{\varkappa}$ with imply the kernel $\overline{K}$ is positive. Fortunately these arguments also let us determine when $y' > 0$ and therefore the comparison results follow from (1.5). The proofs involved are no harder for higher order equations and are given in Section 3.

2. Affine arc length, affine curvature, and the differential equation for the area function.

We give our conventions on the geometry of affine curves. Other sources for this material are [1], [6], [11], and [2]. If $v, w \in \mathbb{R}^2$ then $v \wedge w = v_1w_2 - v_2w_1$ where $v = (v_1, v_2)$ and $w = (w_1, w_2)$. This is the determinant of the matrix with rows v and w. If I is an interval and $c\colon I \to \mathbb{R}^2$ is a C^2 curve, where the velocity vector $c'(t)$ and the acceleration vector $c''(t)$ are linearly independent for all $t \in I$, then a direct calculation shows the 1-form

$$(c'(t) \wedge c''(t))^{1/3}\ dt$$

is invariant under both C^2 reparameterizations and special affine motions of $\mathbb{R}^2$. The ***affine arc length*** of c, denoted $\Lambda(c)$, is defined by integrating this one form. Thus $\Lambda(c) = \int_a^b (c'(t) \wedge c''(t))^{1/3}\,dt$ where $[a, b]$ is the domain of c. Equivalently if $c\colon I \to \mathbb{R}^2$, then s is affine arc length along c if and only if

$$c'(s) \wedge c''(s) \equiv 1.$$

This gives the curve a natural orientation: the orientation for which the basis $c'(s), c''(s)$ is always positive (that is a right handed) along c. This is the same as the orientation that makes the one form $c'(t) \wedge c''(t)^{\frac{1}{3}}\,dt$ positive. The vector $c'(s)$ is the ***affine tangent vector*** and $c''(s)$ is the ***affine normal vector***. With this orientation the Euclidean curvature, κ, is positive which implies c is locally convex in the sense that at any point its tangent line is a locally a support line in that near the point the curve lines in the closed half plane bounded by the tangent line and with the affine normal pointing into the half plane.

Using the product rule to take the derivative of $c'(s) \wedge c''(s) = 1$ and using $c'(s) \wedge c'(s) = 0$ gives

$$c'(s) \wedge c'''(s) = 0.$$

This implies $c'''(s)$ is linearly dependent on $c'(s)$ and therefore for some scalar function $\varkappa(s)$,

$$c'''(s) = -\varkappa(s)c'(s).$$

The function $\varkappa$ is the ***affine curvature*** of c. (This sign is chosen so that ellipses have constant positive affine curvature and hyperbolas have constant negative curvature.) The affine curvature determines a curve up to an affine motion:

Theorem 2.1. *Let I be in interval in $\mathbb{R}$ and $c_1, c_2 \colon I \to \mathbb{R}^2$ affine unit speed curves that are C^3 functions of affine arclength that have the same affine curvature at each point. Then c_1 and c_2 differ by an affine motion. That is for some linear map $M \colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\det(M) = 1$ and some $b \in \mathbb{R}^2$ where holds $c_2(s) = Mc_1(s) + b$ for all $s \in I$.* $\square$

Proofs can be found in [1], [6], and [11]. It is worth remarking that while this theorem only requires $c_1(s)$ and $c_2(s)$ to be C^3 functions of the affine arc length, the images of these curves will be C^4 immersed submanifolds of $\mathbb{R}^2$. See Proposition 5.4 below.

If $c \colon I \to \mathbb{R}^2$ for some interval I, $a \in I$ and $p_0 \in \mathbb{R}^2$, define

$$A_{c,a,p_0}(s) := \begin{cases} \text{Signed area bounded by the restriction } c\big|_{[a,s]} \\ \text{and the segments } \overline{p_0 c(a)} \text{ and } \overline{p_0 c(s)}. \end{cases}$$

See Figure 3. To give a precise, and more computationally useful, parameterize the region in question by the function

$$f(t,s) = (1-t)p_0 + tc(s), \qquad 0 \le t \le 1, \quad s \in I.$$

Then $A_{c,a,p_0}(s)$ is the signed area of the image of f, which is a quantity we can compute with an integral. The partial derivatives and Jacobian of f are

$$\begin{aligned} \frac{\partial f}{\partial t} &= c(s) - p_0 \\ \frac{\partial f}{\partial s} &= tc'(s) \\ \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s} &= t(c(s) - p_0) \wedge c'(s). \end{aligned}$$

So a precise definition of A_{c,a,p_0} is

$$
\begin{aligned}
A_{c,a,p_0}(s) &= \int_a^s \int_0^1 \frac{\partial f}{\partial t}(t,\sigma) \wedge \frac{\partial f}{\partial s}(t,\sigma)\, dt\, d\sigma \\
&= \int_a^s \int_0^1 t(c(\sigma)-p_0) \wedge c'(\sigma)\, dt\, d\sigma \\
&= \frac{1}{2}\int_a^s (c(\sigma)-p_0)\wedge c'(\sigma)\, d\sigma.
\end{aligned} \tag{2.1}
$$

In computing the area $A_{c,a,p_0}(s)$ the points where $f_t \wedge f_u = t(c(\sigma) - p_0) \wedge c'(\sigma)$ is positive the area is counted as positive, and when this is negative the area is negative the area is negative.

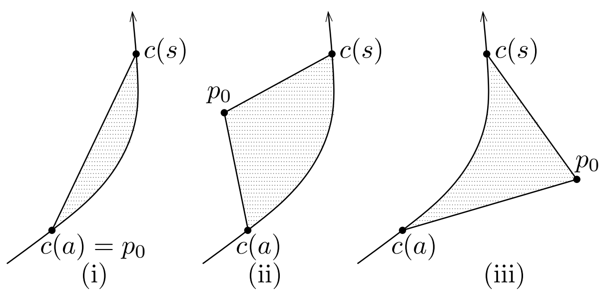

Figure 3. The area $A_{c,a,p_0}(s)$ for some choices of the point p_0. In (i) and (ii) we have $A_{c,a,p_0}(s) > 0$ and in (iii) $A_{c,a,p_0}(s) < 0$.

By taking the first three derivatives of $A(s) := A_{c,a,p_0}(s)$ and using $c'(s)\wedge c''(s) = 1$ and $c'''(s) = -\varkappa(s)c(s)$ we find it satisfies a third order differential equation.

$$
A'(s) = \frac{1}{2}(c(s)-p_0)\wedge c'(s) \tag{2.2}
$$

$$
\begin{aligned}
A''(s) &= \frac{1}{2}c'(s)\wedge c'(s) + \frac{1}{2}(c(s)-p_0)\wedge c''(s) \\
&= \frac{1}{2}(c(s)-p_0)\wedge c''(s)
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
A'''(s) &= \frac{1}{2}c'(s)\wedge c''(s) + \frac{1}{2}(c(s)-p_0)\wedge c'''(s) \\
&= \frac{1}{2} + \frac{1}{2}(c(s)-p_0)\wedge(-\varkappa(s)c'(s)) \\
&= \frac{1}{2} - \varkappa(s)A'(s).
\end{aligned} \tag{2.4}
$$

This proves:

Theorem 2.2. *For any point p_0 the area function $A(s) = A_{c,a,p_0}(s)$ satisfies the third order differential equation*

$$A''' + \varkappa A' = \frac{1}{2}.$$

with initial conditions

$$A(a) = 0, \quad A'(a) = \frac{1}{2}(c(a) - p_0) \wedge c'(a), \quad A''(a) = \frac{1}{2}(c(a) - p_0) \wedge c''(a).$$

□

Corollary 2.3. *Let I be an interval and $c\colon I \to \mathbb{R}^2$ an affine unit speed curve which is a C^3 function of affine arc length. Then for any point $p_0 \in \mathbb{R}^2$ and any $a \in I$ the function $A(s) = A_{c,a,p_0}$ determines c up to an affine motion.*

Proof. In light of Theorem 2.1 it is enough to show A determines the affine curvature, $\varkappa$, of c. As $A'''(s) + \varkappa(s)A'(s) = 1/2$ at points where $A'(s) \neq 0$ we have $\varkappa(s) = (1/2 - A'''(s))/A'(s)$. By continuity of $\varkappa$ to finish it is enough to show the points where $A'(s) = 0$ are isolated in I. If $A'(s) = 0$ and $A''(s) \neq 0$, then s is an isolated zero of A. So it is enough to show there is at most one point where $A'(s) = A''(s) = 0$. As $A'(s) = (1/2)(c(s) - p_0) \wedge c'(s)$ and $A''(s) = (1/2)(c(s) - p_0) \wedge c''(s)$ and the vectors $c'(s)$ and $c''(s)$ are linearly independent $A'(s) = A''(s) = 0$ implies $c(s) = p_0$ and there is at most one such s. □

2.1. **Some notation and conventions.** Some of our results are awkward to state for curves given with a particular parameterization on an interval, that is as $c\colon I \to \mathbb{R}^2$. So we will use $\mathcal{C}$ to denote an embedded convex curve. If $p \in \mathcal{C}$ then $\mathbf{t}_{\mathcal{C}}(p)$ and $\mathbf{n}_{\mathcal{C}}(p)$ will be the affine tangent and affine normal. Explicitly if $c\colon I \to \mathcal{C}$ is a local affine unit speed local parameterization of $\mathcal{C}$ with $c(s_0) = p$, then $\mathbf{t}_{\mathcal{C}}(p) = c'(s_0)$ and $\mathbf{n}_{\mathcal{C}}(s_0) = c''(s_0)$. The affine curvature of $\mathcal{C}$ at p is denoted $\varkappa(p)$.

3. Comparison results for initial value problems.

Definition 3.1. Let I be an interval in $\mathbb{R}$ and let $\mathcal{L}$ be a linear differential operator defined on $C^n(I)$ by

$$\mathcal{L}y = \frac{d^n y}{ds^n} + \sum_{j=0}^{n-1} a_j(s)\frac{d^j y}{ds^j}$$

where the functions $a_0, \ldots, a_{n-1}$ are continuous on I. Then the ***Lagrange kernel*** for $\mathcal{L}$ on this interval is the function $K\colon I \times I \to \mathbb{R}$

defined by the initial value problem

$$\frac{\partial^n K}{\partial s^n}(s;r) + \sum_{j=0}^{n-1} a_j(s) \frac{\partial^j K}{\partial s^j}(s;r) = 0 \tag{3.1}$$

$$\frac{\partial^j K}{\partial s^j}(r;r) = 0 \quad \text{for} \quad 0 \le j \le n-2, \qquad \frac{\partial^{n-1} K}{\partial s^{n-1}}(r;r) = 1. \tag{3.2}$$

One way to see such a kernel exists and is unique to to note that holding r fixed the equation (3.2) is a linear ordinary differential equation for the function $s \mapsto K(s;r)$ so the existence and uniqueness of K follows from the existence and uniqueness theorem for linear ordinary differential equations. The continuity of $K(s;r)$ follows from results on the continuous dependence of solutions of differential equations on initial conditions and parameters (cf. [4, Chapter 1]). More precise regularity for this kernel is given in Remark 3.3.

Remark 3.2. There does not seem to be a standard name for this kernel. Some authors refer to it as a Green's function (e.g. [8]), but as this term is usually reserved for boundary value problems rather than initial value problems this seems a little misleading. As it is used to solve the Cauchy problem for the non-homogeneous $\mathcal{L}y = f$ some authors (e.g. [7]) refer to it as the Cauchy kernel. But as it is just the kernel one gets by applying Lagrange's method of variation of parameters (cf. [8, Pages 145–147] and equation (3.5) below) it seems likely that Lagrange was one of the first to write down some form of it.

Remark 3.3. If $\varphi_1, \varphi_2, \dots, \varphi_n$ is a basis of the solution space to $\mathcal{L}u = 0$, then we can give an explicit formula for K in terms of this basis. As $s \mapsto K(s;r)$ is a solution to $\mathcal{L}y = 0$

$$K(s;r) = \sum_{k=1}^{n} u_k(r)\varphi_k(s) \tag{3.3}$$

for some functions u_k. The initial conditions 3.2 imply

$$\begin{aligned} \frac{\partial^j K}{\partial s^j}(r;r) &= \sum_{k=1}^{n} u_k(r)\varphi_k^{(j)}(r) = 0, \qquad 0 \le j \le n-2 \\ \frac{\partial^{n-1} K}{\partial s^{n-1}}(r;r) &= \sum_{k=1}^{n} u_k(r)\varphi_k^{(n-1)}(r) = 1. \end{aligned} \tag{3.4}$$

Let W be the $n\times n$ matrix with first row $\varphi_1(r),\varphi_2(r),\dots,\varphi_n(r)$ and whose successive rows are the successive derivatives of this row:

$$W=\left[\varphi_k^{(j)}(r)\right]=\begin{bmatrix}\varphi_1(r) & \varphi_2(r) & \cdots & \varphi_n(r)\\ \varphi_1'(r) & \varphi_2'(r) & \cdots & \varphi_n'(r)\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_1^{(n-1)}(r) & \varphi_2^{n-1)}(r) & \cdots & \varphi_n^{(n-1)}(r).\end{bmatrix}$$

Let W_{jk} be the $(n-1)\times(n-1)$ matrix resulting by deleting the j-th row and k-th column of W. Set

$$w=\det(w),\qquad w_{jk}=(-1)^{j+k}\det(W_{jk}).$$

The function w is the Wronskian of the equation and satisfies $w'=-a_{n-1}w$ and therefore does not vanish (cf. [4, page 83]). Then, using Cramer's rule to solve for the $u_k(r)$ in the system (3.4) and using the result in (3.3), we have

$$K(s;r)=\frac{1}{w(r)}\sum_{k=1}^{n}w_{nk}(r)\varphi_k(s).$$

It is worth noting that using Lagrange's method of variation of parameters gives

$$y=\sum_{k=1}^{n}\int_r^s\frac{w_{nk}(t)}{w(t)}\,dr\,\varphi_k(s)\tag{3.5}$$

as the solution to the initial value problem (3.10) of Theorem 3.4. This can be used to give anther, though less direct proof, of Theorem 3.4. For use below we note for $n=2$ and $n=3$ these formulas reduce to

$$K(s;r)=\frac{-\varphi_2(r)\varphi_1(s)+\varphi_1(r)\varphi_2(s)}{w(r)}\tag{3.6}$$

$$K(s;r)=\frac{(\varphi_2(r)\varphi_3'(r)-\varphi_2'(r)\varphi_3(r))\varphi_1(s)}{w(r)}\tag{3.7}$$

$$-\frac{(\varphi_1(r)\varphi_3'(r)-\varphi_1'(r)\varphi_3(r))\varphi_2(s)}{w(r)}\tag{3.8}$$

$$+\frac{(\varphi_1(r)\varphi_2'(r)-\varphi_1'(r)\varphi_2(r))\varphi_3(s)}{w(r)}.\tag{3.9}$$

In general $K(s;r)$ as a function of r depends on $\varphi_k^{(j)}(r)$ for $j=0,1,\dots,(n-2)$. When the coefficients $a_1,a_1,\dots,a_{n-1}$ of $\mathcal{L}$ are only continuous the functions φ_k will only be C^n and therefore $K(s;r)$ is C^2 as a function of r and C^n as a function of s. When the coefficients are C^k then K is a C^{k+2} function of (s,r).

Theorem 3.4. *Let $\mathcal{L}$ be as in Definition 3.1, the function $f\colon I \to \mathbb{R}$ continuous, and $r \in I$. Then, denoting the j-th derivative of y by $y^{(j)}$, the solution to the initial value problem*

$$\mathcal{L}y = f, \qquad y^{(j)}(r) = 0 \quad \textit{for} \quad j = 0, 1, \dots, n-1 \tag{3.10}$$

is

$$y(s) = \int_r^s K(s;t)f(t)\,dt.$$

Proof. This is known (and seems to be a folk theorem among some applied mathematicians). A proof based using variation of parameters can be found in [8, Theorem 4-2, Page 149]. For completeness we include a short proof based directly on the defining properties of K. From the fundamental theorem of calculus and Leibniz's formula for differentiating the integral of a function depending on a parameter:

$$\frac{\partial}{\partial s}\int_r^s \frac{\partial^j K}{\partial s^j}(s;t)f(t)\,dt = \frac{\partial^j K}{\partial s^j}(s;s)f(s) + \int_r^s \frac{\partial^{j+1} K}{\partial s^{j+1}}(s;t)f(t)\,dt \tag{3.11}$$

holds for $j = 0, \dots, (n-1)$. When $j \le n-2$ we have $(\partial^j K/\partial s^j)(s;s) = 0$ and this gives

$$\frac{\partial}{\partial s}\int_r^s \frac{\partial^j K}{\partial s^j}(s;t)f(t)\,dt = \int_r^s \frac{\partial^{j+1} K}{\partial s^{j+1}}(s;t)f(t)\,dt$$

and therefore for $j = 0, 1, \dots, n-1$

$$y^{(j)}(s) = \int_r^s \frac{\partial^j K}{\partial s^j}(s;t)f(t)\,dt.$$

When $j = n-1$, using $(\partial^{n-1}K/\partial s^{n-1})(s;s) = 1)$ and the differential equation for K,

$$\begin{aligned} y^{(n)}(s) &= f(s) + \int_r^s \frac{\partial^n K}{\partial s^n}(s;t)f(t)\,dt \\ &= f(s) + \int_r^s \left(-\sum_{j=0}^{n-1} a_j(s)\frac{\partial^j K}{\partial s^j}(s;t)\right) f(t)\,dt \\ &= f(s) - \sum_{j=0}^{n-1} a_j(s)\int_r^s \frac{\partial^j K}{\partial s^j}(s;t)f(t)\,dt \\ &= f(s) - \sum_{j=0}^{n-1} a_j(s)y^{(j)}(s) \end{aligned}$$

and therefore $\mathcal{L}y = f$. The initial conditions $y^{(j)}(r) = 0$ for $j = 0, 1, , \dots, n-1$ follow from $\int_r^r = 0$. $\square$

Definition 3.5. Let I be an interval and $\mathcal{L}$ a linear differential operator as in Definition 3.1 and let K be the Lagrange kernel of $\mathcal{L}$. Then K is ***forward positive*** on I if and only if for all $r, s \in I$

$$s > r \quad \text{implies} \quad K(s; r) \geq 0.$$

It is ***strongly forward positive*** if and only if $s < r$ implies the function $t \mapsto K(s; t) > 0$ for almost all $t \in (s, r)$.

The definition of a Lagrange kernel being forward positive is motivated by the following comparison result. (This result can be be generalized to a larger class of differential operators, but the statement of the results are awkward to state and more generality is not needed here.)

Theorem 3.6. *Let $\varkappa, \overline{\varkappa} \colon [a, b] \to \mathbb{R}$ be continuous and n and ℓ integers with $0 \leq \ell < n$. Let $y, \overline{y} \colon [a, b] \to \mathbb{R}$ be C^n functions that satisfy the n-order differential equations*

$$y^{(n)} + \varkappa y^{(\ell)} = f \quad \textit{and} \quad \overline{y}^{(n)} + \overline{\varkappa}\,\overline{y}^{(\ell)} = f \tag{3.12}$$

for some continuous function f and have the same initial conditions at a: $y^{(j)}(a) = \overline{y}^{(j)}(a)$ for $j = 0, 1, \dots, n-1$. Assume

(a) The Lagrange kernel, $\overline{K}$, of the differential operator $\overline{y} \mapsto \overline{y}^{(n)} + \overline{\varkappa}\,\overline{y}^{(\ell)}$ is forward positive on $[a, b]$, and

(b) $y^{(\ell)} > 0$ almost everywhere on $[a, b]$.

Then

$$\varkappa \leq \overline{\varkappa} \textit{ on } [a, b] \quad \textit{implies} \quad y \geq \overline{y} \textit{ on } [a, b], \tag{3.13}$$

$$\varkappa \geq \overline{\varkappa} \textit{ on } [a, b] \quad \textit{implies} \quad y \leq \overline{y} \textit{ on } [a, b]. \tag{3.14}$$

If the Lagrange kernel $\overline{K}$ is strongly positive, then $y(b) = \overline{y}(b)$ in either of (3.13) or (3.14) implies $y(s) = \overline{y}(s)$ and $\varkappa(s) = \overline{\varkappa}(s)$ for all $s \in [a, b]$.

Proof. Subtract the first equation in (3.12) form the second to and rearrange a bit to get

$$(\overline{y} - y)^{(n)} + \overline{\varkappa}(y - y)^{(\ell)} = -(\overline{\varkappa} - \varkappa) y^{(\ell)}$$

As y and $\overline{y}$ have the same initial conditions at a the function $\overline{y} - y$ and its first $(n-1)$ derivatives vanish at a. Therefore Theorem 3.4 yields

$$\overline{y}(s) - y(s) = -\int_a^s \overline{K}(s; t)(\overline{\varkappa}(t) - \varkappa(t)) y^{(\ell)}(t)\, dt.$$

By our assumptions for each s the inequality $K(s; t) y^{(\ell)}(t) \geq 0$ for all $t \in [s, b]$. Thus if $\varkappa \leq \overline{\varkappa}$ on $[a, b]$ we have $y \geq \overline{y}$ on $[a, b]$. If $\overline{K}$ is is strongly positive then $\overline{K}(s; t) y^{(j)}(t) > 0$ for almost all $t \in [s, b]$.

Therefore if $y(b) = \overline{y}(b)$, then $\overline{\varkappa}(t) = \varkappa(t)$ for $t \in [a, b]$ which implies y and $\overline{y}$ stratify the same initial value problem on $[a, b]$ and thus are equal on this interval. A similar argument holds if $\varkappa \geq \overline{\varkappa}$ on $[a, b]$. $\square$

Proposition 3.7. *Let $\mathcal{L}$ be a differential operator $y \mapsto y^{(n)} + \sum_{j=0}^{n-1} a_j y^{(j)}$ with Lagrange kernel K. Then K being forward positive implies that K is strongly forward positive in the following cases:*

(1) $\mathcal{L}$ is constant coefficient,
(2) $n = 2$, or
(3) $n = 3$ and a_1 and a_2 are continuously differentiable.

Proof. In all three cases we show that the for fixed s the function $r \mapsto K(s; r)$ satisfies a linear homogeneous differential equation. The zeros of such a function are isolated and therefore it has at most countable many zeros. Therefore if it is also nonnegative it is positive except at countable many points and thus is positive almost everywhere.

In the case $\mathcal{L}$ has constant coefficients let φ be the solution with $\varphi^{(j)}(0) = 0$ for $0 \leq j \leq n - 2$ and $\varphi^{(n-1)}(0) = 1$. Then the Lagrange kernel is $K(s; r) = \varphi(s - r)$ and thus $r \mapsto K(s; r)$ is a solution to $y \mapsto y^{(n)} + \sum_{j=0}^{n-1} (-1)^{n-j} y^{(j)} = 0$.

When $n = 2$ the form of K is given by (3.6) and it follows that

$$\frac{\partial^2 (w(r)K(s; r))}{\partial r^2} + a_1(r) \frac{\partial (w(r)K(s; r))}{\partial r} + a_0(r)(w(r)K(s; r)) = 0.$$

Thus $r \mapsto K(s; r)$ satisfies a second order linear differential equation.

When $n = 3$ and a_1 and a_2 are C^1 the form of $K(s; r)$ is given by (3.9). Let ψ_1 and ψ_2 be solutions to $\mathcal{L}(y) = 0$ and set $u = \psi_1 \psi_2' - \psi_1' \psi_2$. Then a direct, but tedious, calculation yields

$$u''' + a_2 u'' + (a_1 + a_2 + a_2')u' + (a_1' - a_0)u = 0. \tag{3.15}$$

From (3.9) we see for fixed s the function $r \mapsto w(r)K(s; r)$ is a linear combination of terms of this form and so it also satisfies the differential equation (3.15). Thus $r \mapsto K(s; r)$ has a most countably zeros on any interval and so if it is nonnegative on an interval, it is positive almost everywhere on the interval. $\square$

Remark 3.8. I conjecture that in all cases that a kernel being forward positive implies that it is strongly forward positive and that even when $n = 3$ the extra smoothness on the coefficients is not required.

To give the Lagrange kernel for the constant coefficient linear operators related to our problem it is convenient to introduce some notation. Let k be a real number and define functions $\mathbf{c}_k$ and $\mathbf{s}_k$ by the initial value problems

$$\mathbf{c}_k'' + k\mathbf{c}_k = 0, \qquad \mathbf{c}_k'(0) = 1, \quad \mathbf{c}_k'(0) = 0$$
$$\mathbf{s}_k'' + k\mathbf{s}_k = 0, \qquad \mathbf{s}_k(0) = 0, \quad \mathbf{s}_k'(0) = 1$$

or more explicitly

$$\mathbf{c}_k(s) = \begin{cases} \cos(\sqrt{k}\, s), & k > 0; \\ 1, & k = 0; \\ \cosh(\sqrt{|k|}\, s), & k < 0. \end{cases} \quad \mathbf{s}_k(s) = \begin{cases} \dfrac{\sin(\sqrt{k}\, s)}{\sqrt{k}}, & k > 0; \\ s, & k = 0; \\ \dfrac{\sinh(\sqrt{|k|}\, s)}{\sqrt{|k|}}, & k < 0. \end{cases}$$

These satisfy

$$\mathbf{c}_k' = -k\mathbf{s}_k, \qquad \mathbf{s}' = \mathbf{c}_k \tag{3.16}$$
$$\mathbf{c}_k^2 + k\mathbf{s}_k^2 = 1 \tag{3.17}$$
$$\mathbf{c}_k(a+s) = \mathbf{c}_k(a)\mathbf{c}_k(s) - k\mathbf{s}_k(a)\mathbf{s}_k(s) \tag{3.18}$$
$$\mathbf{s}_k(a+s) = \mathbf{s}_k(a)\mathbf{c}_k(s) + \mathbf{c}_k(a)\mathbf{s}_k(s). \tag{3.19}$$

Possibly the easiest way to see the derivative formulas hold is to note $\mathbf{c}_k'$ and $-k\mathbf{s}_k$ are both solutions to the initial value problem $y'' + ky = 0$, $y(0) = 0$, $y'(0) = -k$, and $\mathbf{s}_k'$ and $\mathbf{c}_k$ are solutions to $y'' + ky = 0$, $y(0) = 1$ and $y'(0) = 0$. These imply $\mathbf{c}_k^2 + k\mathbf{s}_k^2$ has zero as its derivative and therefore is constant, this implies (3.17) holds. For the addition formula for $\mathbf{c}_k$, note the left and right hand sides of (3.18) both satisfy the initial value problem $u''+ku = 0$, $u(0) = \mathbf{c}_k(a)$ and $u'(0) = \mathbf{c}_k'(a) = -k\mathbf{s}_k(a)$. A similar argument shows the addition formula for $\mathbf{s}_k$ holds.

Proposition 3.9. *Let $k \in \mathbb{R}$ and I an interval.*

(a) The Lagrange kernel for $y \mapsto y'' + ky$ on I is

$$P_k(s; r) = \mathbf{s}_k(s - r).$$

When $k \leq 0$, this is strongly forward positive on all intervals I. When $k > 0$ this is strongly forward positive on all intervals of length L satisfying $k \leq (\pi/L)^2$.

(b) The Lagrange kernel for $y \mapsto y''' + ky'$ is

$$Q_k(s; r) = \begin{cases} \dfrac{1 - \mathbf{c}_k(s-r)}{k}, & k \neq 0; \\ \dfrac{(s-r)^2}{2}, & k = 0. \end{cases}$$

and this is strongly forward positive on all intervals for all k.

Proof. A direct calculation using (3.16) shows these functions satisfy the conditions defining the Lagrange kernel and it is straightforward to check when they are strongly forward positive. □

Lemma 3.10. *Let* $\varkappa\colon [a,b] \to \mathbb{R}$ *be continuous and let* $u \in C^2([a,b])$ *satisfy* $u'' + \varkappa u = 0$.

(a) If $u(a) = 0$, $u'(a) \neq 0$, *and* $k_0 \leq (\pi/(b-a))^2$, *then* $u \neq 0$ *on* (a,b).

(b) If $u(a) \neq 0$, $u'(a) = 0$, *and* $k_0 \leq (\pi/(2(b-a))^2$, *then* $u \neq 0$ *on* (a,b).

Proof. While (a) can be proven using the Sturm Comparison Theorem, [4, Theorem 1.1 Page 208], we give a short proof based on Theorem 3.6 which has the advantage of working for (b) as well. Without loss of generality we may assume $u'(a) = 1$, in which case $u(s) > 0$ for $s > a$ and s near a. Towards a contradiction assume u has a zero in (a,b) and let b^* be the smallest zero of u in (a,b). Then $u > 0$ on (a,b^*). Let $\overline{u}(s) := \mathbf{s}_{k_0}(s-a)$. Then

$$\overline{u}'' + k_0\overline{u} = 0, \qquad \overline{u}(a) = 0, \quad \overline{u}'(a) = 1.$$

By Proposition 3.9 the Lagrange kernel for $\overline{u} \mapsto \overline{u}'' + k_0\overline{u}$ is $\overline{K}(s;r) = P_{k_0}(s;r) = \mathbf{s}_{k_0}(s-r)$ and this is forward positive on all intervals (when $k_0 \leq 0$) or on any interval whose length satisfies $k_0 \leq (\pi/L)^2$ (when $k_0 > 0$). Therefore by Theorem 3.6 (with $n = 2$ and $\ell = 0$) we have $\overline{u}(s) = \mathbf{s}_{k_0}(s-a) \leq u(s)$ on $(0,b^*)$. As $u(b^*) = 0$, this implies $\mathbf{s}_{k_0}(b^*-a) \leq 0$ so that $\mathbf{s}_{k_0}(s-a) = 0$ would have a solution in $(a,b^*]$, which is not the case.

For (b) the same idea works, use 3.6 to compare u to $\overline{u} := \mathbf{c}_{k_0}$. □

Theorem 3.11. *Let* I *be an interval and* $\varkappa\colon I \to \mathbb{R}$ *continuous. Then the Lagrange kernel for* $y \mapsto y''' + \varkappa y'$ *is forward positive in the following cases:*

(a) $\varkappa$ *is constant,*

(b) $\varkappa \leq 0$, *or*

(c) $\varkappa \leq k_1$ *for a positive constant* k_1 *with* $k_1 \leq (\pi/L)^2$ *where* L *is the length of* I.

In case (a) the Lagrange is always strongly forward positive. In cases (b) and (c) if $\varkappa$ *is* C^1, *then the Lagrange kernel is strongly forward positive.*

Proof. The case of $\varkappa$ being constant follows form Proposition 3.9.

The hypothesis of case (b) can be restated as $\varkappa \leq k_1 := 0$. With this notation we have $\varkappa \leq k_1$ in both cases (b) and (c). Let let $y(s) := K(s;r)$, then $(y')'' + \varkappa(y') = 0$, $y'(a) = 0$, $(y')'(a) = 1$. Applying Lemma 3.10 to $u := y'$ yields $y' > 0$ on $I \cap (r,\infty)$ in both cases (b) and (c). Let $\overline{K}$ be the Lagrange kernel for the $\overline{y} \mapsto \overline{y}''' + k_1\overline{y}'$. Then $\overline{K}$ is forward positive by case (a). Theorem 3.6 (with $n = 3$ and $\ell = 1$) implies $y \geq \overline{y}$ on $I \cap (r, \infty)$. When $k_1 = 0$ we have $\overline{y}(s) = (s-r)^2/2 > 0$ and when $k_1 > 0$ we have $\overline{y}(s) = (1 - \mathbf{c}_{k_1}(s-r))/k_1 \geq 0$. As $y(s) = K(s; r)$ and r was any element of I we are done.

The statements about strong forward positivity follow from Proposition 3.7. □

Lemma 3.12. *Let $k \in \mathbb{R}$, then the solutions to the initial value problems*

$$\begin{aligned} &\overline{x}_k''' + k\overline{x}_k' = 0, \qquad \overline{x}_k(0) = 0, \quad \overline{x}_k'(0) = 1, \quad \overline{x}_k''(0) = 0 \\ &\overline{y}_k''' + k\overline{y}_k' = 0, \qquad \overline{y}_k(0) = 0, \quad \overline{y}_k'(0) = 0, \quad \overline{y}_k''(0) = 1 \end{aligned}$$

are

$$\overline{x}_k(s) = \mathbf{s}_k(s)$$
$$\overline{y}_k(s) = \begin{cases} \dfrac{1 - \mathbf{c}_k(s)}{k}, & k \neq 0 \\ \dfrac{s^2}{2}, & k = 0. \end{cases}$$

The curve $\overline{c}_k(s) := (\overline{x}_k(s), \overline{y}_k(s))$ has unit affine speed and constant affine curvature k. This parameterizes the connected component containing $(0,0)$ of the conic with equation

$$x^2 + ky^2 - 2y = 0.$$

Proof. Direct calculations using equations (3.16) and (3.17). □

Proposition 3.13. *Let $\varkappa$ be continuous on $[a, b]$ and let $x \in C^3([a, b])$ satisfy*

$$x''' + \varkappa x' = 0, \qquad x(a) = 0, \quad x'(a) = 1, \quad x''(a) = 0.$$

Assume for some constants k_0 and k_1 that $k_0 \leq \varkappa \leq k_1$ and $k_1 \leq (\pi/(2(b-a)))^2$. Then, with the notation of Lemma 3.12, the inequalities

$$\overline{x}_{k_1}(b-a) \leq x(b) \leq \overline{x}_{k_0}(b-a)$$

hold. If equality holds in the lower bound (respectively in the upper bound), then $\varkappa(s) = k_1$ and $x(s) = \overline{x}_{k_1}(s-a)$ (respectively $\varkappa = k_0$ and $x(s) = \overline{x}_{k_0}(s-a)$) on $[a, b]$.

Proposition 3.14. *Let $\varkappa$ be continuous on $[a, b]$ and let $y \in C^3([a, b])$ satisfy*

$$y''' + \varkappa y' = 0, \qquad y(a) = 0, \quad y'(a) = 0, \quad y''(a) = 1.$$

Assume for some constants k_0 and k_1 that $k_0 \leq \varkappa \leq k_1$ and $k_1 \leq (\pi/(b-a))^2$. Then, with the notation of Lemma 3.12, the inequalities

$$\overline{y}_{k_1}(b-a) \leq y(b) \leq \overline{y}_{k_0}(b-a)$$

hold. If equality holds in the lower bound (respectively in the upper bound), then $\varkappa(s) = k_1$ and $y(s) = \overline{y}_{k_1}(s-a)$ (respectively $\varkappa = k_0$ and $y(s) = \overline{y}_{k_0}(s-a)$) on $[a,b]$.

Proof of Propositions 3.13 and 3.14. We prove Proposition 3.13, the proof of Proposition 3.14 being similar. By Part (b) of Lemma 3.10 applied to the function $u = x'$ we see that $x' > 0$ on (a,b). Therefore the lower bound on x follows form Theorem 3.6 by comparing x to $\overline{x}(s) = \overline{x}_{k_1}(s)$ and the upper bound by comparing to $\overline{x}(s) := \overline{x}_{k_0}(s)$. Theorem 3.6 also covers the cases of equality. □

4. Comparisons for areas

Theorem 4.1. *Let D be a convex open set in $\mathbb{R}^2$ (which need not be bounded) with C^4 boundary. Let $p_0 \in \partial D$ and let $c\colon [0,L] \to \partial D$ have affine unit speed and $c(0) = p_0$ and let $\varkappa$ be the affine curvature of c. Let $\overline{\varkappa}\colon [0,L] \to \mathbb{R}$ be a continuous function so that the operator $\overline{y} \mapsto \overline{y}''' + \overline{\varkappa}\,\overline{y}'$ has forward positive Lagrange kernel and define $\overline{A}\colon [0,L] \to \mathbb{R}$ by the initial value problem*

$$\overline{A}''' + \overline{\varkappa}\overline{A}' = \frac{1}{2}, \qquad \overline{A}(0) = \overline{A}'(0) = \overline{A}''(0) = 0$$

Then, with $A_{c,0,c(0)}$ as in (2.1),

$$\varkappa(s) \leq \overline{\varkappa}(s) \ \text{on } [0,L] \quad \text{implies} \quad A_{c,0,c(0)}(L) \geq \overline{A}(L)$$
$$\varkappa(s) \geq \overline{\varkappa}(s) \ \text{on } [0,L] \quad \text{implies} \quad A_{c,0,c(0)}(L) \leq \overline{A}(L)$$

If the Lagrange kernel for $\overline{y} \mapsto \overline{y}''' + \overline{\varkappa}\,\overline{y}'$ is strongly forward positive (which will be the case when $\overline{\varkappa}$ is C^1) then equality in either case implies $\varkappa \equiv \overline{\varkappa}$ on $[0,L]$.

Proof. To simplify notation let $A = A_{c,0,c(0)}$. By Theorem 2.2 A satisfies the differential equation $A''' + \varkappa A = 1/2$ and $p_0 = c(0)$ the formulas (2.1) and (2.2) imply $A(0) = A'(0) = A''(0) = 0$. From Equation 2.2, $A'(s) = \frac{1}{2}(c(s) - c(0)) \wedge c'(s)$. This implies, see Figure 4, $A'(s) > 0$ all s with $c(s) \neq p_0$. Therefore the result follows from Theorem 3.6. □

Let $k \in \mathbb{R}$ and set

$$\overline{A}_k(s) = \begin{cases} \dfrac{s - \mathbf{s}_k(s)}{2k}, & k \neq 0; \\ \dfrac{s^3}{12}, & k = 0. \end{cases} \tag{4.1}$$

Using the definition of $\mathbf{s}_k$ it is not hard to check

$$\overline{A}_k'''(s) + k\overline{A}_k'(s) = \frac{1}{2} \qquad \overline{A}_k(0) = \overline{A}_k'(0) = \overline{A}_k''(0) = 0.$$

With the notation of Theorem 4.1, $A_k = A_{c,0,c(0)}$ where c is a curve with constant affine curvature k.

Corollary 4.2. *With notation as in Theorem 4.1 and Equation 4.1 if* $k_0 \le \varkappa \le k_1$ *for some constants* k_0 *and* k_1*, then*

$$\overline{A}_{k_1}(L) \le A_{c,0,c(0)}(s) \le \overline{A}_{k_0}(L)$$

on the interval $[0, L]$*. If* $A_{c,0,c(0)}(L) = \overline{A}_{k_0}(L)$ *(respectively* $A_{c,0,c(0)}(L) = \overline{A}_{k_1}(L)$*) then* c *has constant curvature* k_0 *(respectively* k_1*) on the interval* $[0, L]$. □

Proposition 4.3. *Let* $\mathcal{C}$ *be convex and have an affine curvature bound* $\varkappa \ge k_0$*. Then any triangle* $\triangle p_1p_2p_3$ *with vertices on* $\mathcal{C}$ *satisfies*

$$\text{Area}(\triangle p_1p_2p_3) < \overline{A}_{k_0}(\Lambda(\mathcal{C})) \tag{4.2}$$

where $\overline{A}_{k_0}$ *as in Equation* (4.1) *and* $\Lambda(\mathcal{C})$ *is the affine length of* $\mathcal{C}$*.*

Proof. Let k_1 be any upper bound for the affine curvature of c. Then the result follows from Corollary 4.2. See Figure 5 □

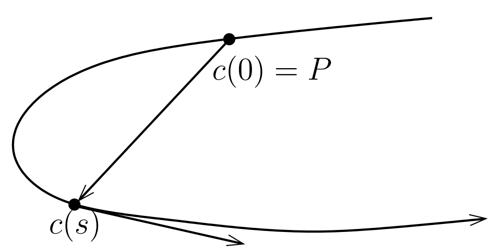

Figure 4. The vectors $c(s) - c(0)$ and $c'(s)$ form a right handed (i.e. positive) basis of $\mathbb{R}^2$.

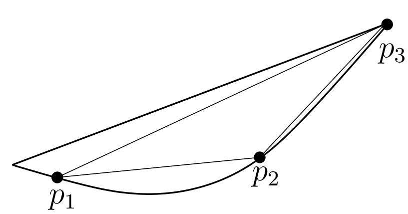

Figure 5. The area of $\triangle p_1p_2p_3$ is less than the area of the region bounded by $\mathcal{C}$ and the segment between the endpoints of $\mathcal{C}$ and this region has area at most $\overline{A}_{k_0}(\Lambda(\mathcal{C}))$ by Corollary 4.2.

Remark 4.4. The estimate in Proposition 4.3 is close to sharp in the sense that for a fixed affine length $\Lambda(\mathcal{C})$ as $k_0 \to -\infty$ the ratio of the two sides of inequality 4.2 goes to 1. To see this let $L > 0$ and $\mathcal{C}$ be the curve parameterized by $\overline{c}_{k_0} \colon [-L, L] \to \mathbb{R}^2$ as in Lemma 3.12. Then, as $\overline{c}_{k_0}$ has affine unit speed, $\Lambda(\mathcal{C}) = 2L$. Let p_1, p_2, and p_3 be the as in Figure 6. Then a calculation using that $\mathbf{s}_{k_0}(2L) = 2\mathbf{c}_{k_0}(L)\mathbf{s}_{k_0}(L)$ (which follows from the addition formula (3.19))

$$\begin{aligned}\frac{\operatorname{Area}(\triangle p_1p_2p_2)}{\overline{A}_{k_0}(\Lambda(\mathcal{C}))} - 1 &= -\frac{\mathbf{s}_{k_0}(L) - L}{\mathbf{s}_{k_0}(L)\mathbf{c}_{k_0}(L) - L} \\ &= -\frac{\sinh(\sqrt{|k_0|}L) - \sqrt{|k_0|}L}{\sinh(\sqrt{|k_0|}L)\cosh(\sqrt{|k_0|}L) - \sqrt{|k_0|}L} \\ &\sim \frac{-e^{-\sqrt{|k_0|}L}}{2} = \frac{-e^{-\sqrt{|k_0|}\Lambda(\mathcal{C})/2}}{2}.\end{aligned}$$

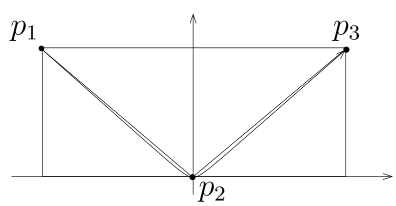

Figure 6. Here $p_1 = (\overline{x}_{k_0}(-L), \overline{y}_{k_0}(-L))$, $p_2 = (0,0)$, and $p_3 = (\overline{x}_{k_0}(L), \overline{y}_{k_0}(L))$. Then $\operatorname{Area}(\triangle p_1p_2p_3) = \overline{x}_{k_0}(L)\overline{y}_{k_0}(L)$. The curve is the hyperbola with equation $x^2 + k_0y^2 - 2y = 0$.

5. Adapted coordinates and geometric bounds.

Definition 5.1. Let $\mathcal{C}$ be a C^2 embedded convex curve and $p_0 \in \mathcal{C}$. Let $I_{\mathcal{C},p_0} \subseteq \mathbb{R}$ be the unique interval so that there is an affine unit speed parameterization $c_{\mathcal{C},p_0} \colon I_{\mathcal{C},p_0} \to \mathcal{C}$ with $c_{\mathcal{C},p_0}(0) = p_0$ that parameterizes all of $\mathcal{C}$. This is the ***standard parameterization*** of $\mathcal{C}$ at p_0. Then the ***affine adapted coordinates*** to $\mathcal{C}$ at p_0 are the linear coordinates ξ, η on $\mathbb{R}^n$ centered at p_0 with

$$\left.\frac{\partial}{\partial\xi}\right|_{p_0} = \mathbf{t}_{\mathcal{C}}(p_0) = c'_{\mathcal{C},p_0}(0), \qquad \left.\frac{\partial}{\partial\eta}\right|_{p_0} = \mathbf{n}_{\mathcal{C}}(p_0) = c''_{\mathcal{C},p_0}(0).$$

The ***graphing parameter set*** is the maximum interval $I^*_{\mathcal{C},p_0} \subseteq I_{\mathcal{C},p_0}$ so that the image of restriction $c_{\mathcal{C},p_0}\big|_{I^*_{\mathcal{C},p_0}} : I^*_{\mathcal{C},p_0} \to \mathcal{C}$ is a graph in the adapted coordinates ξ, η. The ***graphing interval***, $I^{**}_{\mathcal{C},p_0}$, is the set $I^{**}_{\mathcal{C},p_0} := \{\xi(c(s)) : s \in I^*_{\mathcal{C},p_0}\}$. See Figure 7.

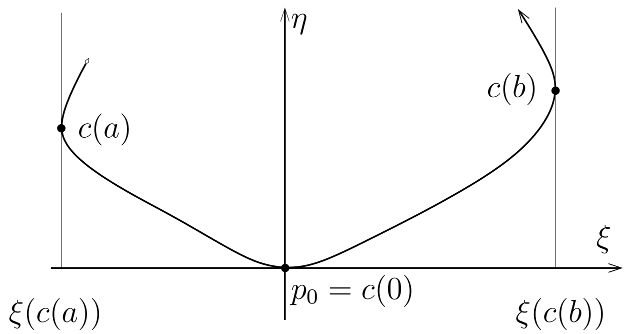

Figure 7. Here $c := c_{\mathcal{C},p_0}$ is the *standard parameterization* of $\mathcal{C}$ at p_0. The affine adapted coordinates centered at $p_0 = c(0)$ have the ξ-axis tangent to $\mathcal{C}$ at p_0 and the η-axis is in the direction of the affine normal $c''(0) = \mathbf{n}_{\mathcal{C}}(0)$. By local convexity the curve is locally the graph of a convex function. Let $[a, b]$ be the maximum subinterval of $I_{\mathcal{C},p_0}$ with $0 \in [a, b]$ so that the restriction $c\big|_{[a,b]}$ is is a graph. This is $I^*_{\mathcal{C},p_0}$, the *graphing parameter set*. The interval $[\xi(c(a)), \eta(c(b))]$ on the ξ-axis which is the domain of the graphing function is the *graphing interval* $I^{**}_{\mathcal{C},p_0}$. Note that the endpoints of $I^{**}_{\mathcal{C},p_0}$ are (when not an endpoint of $I_{\mathcal{C},p_0}$) are the points where the tangent $\mathcal{C}$ become vertical, that is if $x(s) = \xi(c(s))$, the points where $x'(s) = 0$.

Lemma 5.2. *Let $p_0 \in \mathcal{C}$. With the notation of Lemma 5.3 the connected component of* 0 *in* $\{s \in I_{\mathcal{C},p_0} : x'(s) > 0\}$ *is contained in the graphing parameter set, $I^*_{\mathcal{C},p_0}$.*

Proof. Let I_0 be connected component of 0 in $\{s \in I_{\mathcal{C},p_0} : x'(s) > 0\}$ and set $J := \{x(s) : s \in I_0\}$. Then, as $x'(s) > 0$ for I_0, the map $s \mapsto x(s)$ is a diffeomorphism between I_0 and J. Define $f\colon J \to \mathbb{R}$ by

$$f(x(s)) := \int_0^s y'(t)\, dt. \tag{5.1}$$

Taking the derivative of this gives $\frac{d}{ds}\left(f(x(s)) - y(s)\right) = 0$. Therefore $f(x(s)) - y(s) = C$ for some constant C. As $x(0) = y(0) = f(0) = 0$ we have $C = 0$, which implies $y(s) = f(x(s))$ for $s \in I_0$. Therefore I_0 is contained in the graphing parameter set about p_0. $\square$

Lemma 5.3. *Let ξ, η be affine adapted coordinates to $\mathcal{C}$ at p_0 and let $x, y\colon I_{\mathcal{C},p_0} \to \mathbb{R}$ be the coordinates of c in this coordinate system, that is $x(s) := \xi(c(s))$ and $y(s) := \eta(c(s))$ and assume x and y are C^3 functions of affine arclength. Then x and y satisfy the initial value* *problems*

$$\begin{aligned} &x''' + \varkappa x = 0, && x(0) = 0 && x'(0) = 1 && x''(0) = 0 \\ &y''' + \varkappa y = 0, && y(0) = 0 && y'(0) = 0 && y''(0) = 1 \end{aligned}$$

where $\varkappa$ is the affine curvature of $\mathcal{C}$.

Proof. This follows easily from the equation $c''' + \varkappa c' = 0$ and the definition of the adapted coordinates. □

In the definition of the affine adapted coordinates it is assumed the curve $\mathcal{C}$ is of differentiability class C^2. In general the standard parameterization will only be C^2 of affine arclength. If the standard parameterization is C^3 as a function of affine arclength, which is what is required for the affine curvature to be defined, there is a gain in the regularity of $\mathcal{C}$: it will be a C^4 immersed submanifold of $\mathbb{R}^2$.

Proposition 5.4. *With notation as in Lemma 5.2 if $\varkappa$ is a C^k function of affine arclength for $k \geq 0$, then the function f is C^{k+4}. Thus the curve $\mathcal{C}$ is a C^{k+4} immersed submanifold of $\mathbb{R}^2$.*

Proof. As $y(s) = f(x(s))$ and x and y are C^3 it follows f is C^3. Taking three derivatives of (5.1) gives

$$\begin{aligned} y'(s) &= f'(x(s))x'(s) \\ y''(s) &= f''(x(s))x'(s)^2 + f'(x(s))x''(s) \\ y'''(s) &= f'''(x(s))x'(s)^3 + 3f''(x(s))x'(s)x''(s) + f'(x(s))x'''(s). \end{aligned} \tag{5.2}$$

Using $x'''(s) = -\varkappa(s)x'(s)$, $y'''(s) = -\varkappa(s)y'(s)$ and $y'(s) = f'(x(s))x'(s)$ in (5.2) gives

$$0 = f'''(x(s))x'(s)^3 + 3f''(x(s))x'(s)x''(s).$$

The function x' does not vanish on the interior of the graphing parametrization set so we can divide by $x'(s)^3$ to get

$$f'''(x(s)) = -\frac{3f''(x(s))x''(s)}{x'(s)^2}.$$

As x is C^3, and $x' \neq 0$ it has a C^3 inverse g, that there is a C^3 function g with $g(x(s)) = s$. Therefore

$$f'''(x) = -\frac{3f''(x)x''(g(x))}{x'(g(x))^2}$$

which shows f''' is C^1 and therefore f is C^4. For a non-parametric curve of class C^4 given as a graph $y = f(x)$ the affine curvature is (cf.

[1, Page 14, eqn. (83)])

$$\varkappa(x) = -\frac{1}{2}\left(\frac{1}{(f''(x)^{\frac{2}{3}}}\right)'' = \frac{f''''(x)}{2f''(x)^{\frac{5}{3}}} - \frac{5f'''(x)^2}{9f''(x)^{\frac{8}{3}}}$$

which can be rewritten in the form

$$f''''(x) = 2\kappa(x)f''(x)^{\frac{5}{3}} + \frac{10}{9}\frac{f'''(x)^2}{f''(x)}$$

By standard regularity theorems for ordinary differential equations if $\varkappa$ is C^k then f is C^{k+4}. Or one can just take repeated derivatives of this equation and use induction to get the result. □

Theorem 5.5. *Let $\mathcal{C}$ be C^4 and let $p_0 \in \mathcal{C}$. Assume $(-L, L) \subseteq I_{\mathcal{C},p_0}$ for some $L > 0$. Also assume for some constants k_0 and k_1 the affine curvature of $\mathcal{C}$ satisfies the bounds $k_0 \le \varkappa(s) \le k_1$ for $-L \le s \le L$ and $k_1 \le (\pi/2L)^2$. Let ξ, η be affine adapted coordinates at p_0 let $x(s) = \xi(c(s))$ and $y(s) = \eta(c(s))$. Then $(-L, L)$ is contained in the graphing parameter set of $\mathcal{C}$ about p_0. With the notation of Lemma 3.12 the inequalities*

$$\overline{x}_{k_1}(|s|) \le |x(s)| \le \overline{x}_{k_0}(|s|) \tag{5.3}$$

$$\overline{y}_{k_1}(s) \le y(s) \le \overline{y}_{k_0}(s) \tag{5.4}$$

hold on the interval $(-L, L)$. Letting $R := \mathbf{s}_{k_1}(L)$, the graphing interval of $\mathcal{C}$ at s_0 contains $(-R, R)$. If $s_1 \in (-L, L)$ with $s_1 \neq 0$ and equality holds in either of the lower bounds of (5.3) *or* (5.4) *then the restriction of c to the interval of points between* 0 *and s_1 is a curve of constant curvature k_1. Likewise if equality holds in either of the upper bounds of* (5.3) *or* (5.4) *then the restriction of c to the interval of points between* 0 *and s_1 is a curve of constant curvature k_0.*

Proof. Applying Lemma 3.10 to the function $u = x'$ on $[0, L)$ yields $x' > 0$ on $[0, L)$. Applying the same lemma to $u(s) = x'(-s)$ on $[0, L)$ (which satisfies $u''(s) + \varkappa(-s)u(s) = 0$) implies $x' \neq 0$ the interval $(-L, L)$. Therefore $x' > 0$ on $(-L, L)$, thus by Lemma 5.2 the interval $(-L, L)$ is contained in the graphing parameter set of $\mathcal{C}$ at p_0. On the interval $[0, L)$ the inequalities 5.3 and 5.4 on the interval $[0, L)$ follow from Propositions 3.13 and 3.14.

To get the inequalities for $s \in (-L, 0]$ let $\widetilde{x}, \widetilde{y} \colon [0, -L) \to \mathbb{R}$ be given by

$$\widetilde{x}(s) = -x(-s), \qquad \widetilde{y}(s) = y(-s).$$

These satisfy

$$\begin{aligned} &\widetilde{x}'''(s) + \varkappa(-s)\widetilde{x}'(s) = 0, \qquad \widetilde{x}(0) = 0, \quad \widetilde{x}'(0) = 1, \quad \widetilde{x}''(0) = 0, \\ &\widetilde{y}'''(s) + \varkappa(-s)\widetilde{y}'(s) = 0, \qquad \widetilde{y}(0) = 0, \quad \widetilde{y}'(0) = 0, \quad \widetilde{y}''(0) = 1. \end{aligned}$$

Again using Propositions 3.13 and 3.14 along with the definitions of $\widetilde{x}$ and $\widetilde{y}$ gives

$$\begin{aligned} \overline{x}_{k_1}(s) \le -x(-s) \le \overline{x}_{k_0}(s) \\ \overline{y}_{k_1}(s) \le y(-s) \le \overline{y}_{k_0}(s) \end{aligned}$$

on $[0, L)$. Replacing s by $-s$ and using $\overline{x}_{k_j}(-s) = -\overline{x}_{k_j}(s)$ and $\overline{y}_{k_j}(-s) = \overline{y}_{k_j}(s)$ shows the inequalities (5.3) and (5.4) also hold on $(-L, 0]$. The bound (5.3) implies the graphing interval contains $(-R, R)$.

The statements about when equality holds follow from the equality cases in Propositions 3.13 and 3.14. □

Corollary 5.6. *Let $\mathcal{C}$ have the affine curvature bound $\varkappa \le 0$ and assume there is an affine unit speed parameterization $c\colon \mathbb{R} \to \mathcal{C}$ defined on all of $\mathbb{R}$. Then c is bijective and for any point $p_0 \in \mathcal{C}$ the curve is globally a graph $\eta = f(\xi)$ in the affine adapted coordinates at p_0.*

Proof. Let $p_0 \in \mathcal{C}$. Without loss of generality we can assume $c(0) = p_0$. Let $L > 0$. In Theorem 5.5 let $k_0 = \min\{\varkappa(s) : s \in [-L, L]\}$ and $k_1 = 0$ to see the graphing parameter set contains $(-L, L)$ and the graphing interval contains $(-s_{k_1}(L), s_{k_1}(L)) = (-L, L)$ (as $\mathbf{s}_{k_1}(s) = \mathbf{s}_0(s) = s$). Letting $L \to \infty$ finishes the proof. □

Example 5.7. To see that an upper bound on the affine curvature is necessary in this corollary note for any $k > 0$, let $\mathcal{C}$ be the circle with equation $x^2 + y^2 = k^{-3/4}$. Then $c\colon \mathbb{R} \to \mathcal{C}$ given by

$$c(s) = \big(k^{-3/4}\cos(k^{1/2}s), k^{-3/4}\sin(k^{1/2}s)\big)$$

is unit affine speed and $\mathcal{C}$ has constant curvature k, but c is not injective and $\mathcal{C}$ is not globally a graph in any coordinate system.

Theorem 5.8. *Let $\mathcal{C}$ have curvature bounds $k_0 \le \varkappa \le k_1$ with k_0, k_1 constants with $k_1 \le (\pi/\Lambda(\mathcal{C}))^2$. Let p_1, p_2, p_3 be distant points on $\mathcal{C}$. Let $L = \Lambda(\mathcal{C})/2$. Then*

$$\operatorname{Area}(\triangle p_1p_2p_3) \le \overline{x}_{k_0}(L)\overline{y}_{k_0}(L).$$

Equality holds if and only if $\mathcal{C}$ has constant curvature k_0 and, after a possible reordering, the points p_1, p_2, and p_3 are the initial point, midpoint, and endpoint of $\mathcal{C}$.

Proof. Let p_0 be the midpoint of $\mathcal{C}$ and construct the affine adapted coordinates ξ, η for $\mathcal{C}$ centered at p_0. Label the points p_1, p_2 and p_3 so that they are increasing order along $\mathcal{C}$. By Theorem 5.5 the curve lies inside the rectangle defined by $-\overline{x}_{k_0}(L) \le \xi \le \overline{x}_{k_0}(L)$ and $0 \le \eta \le \overline{y}_{k_0}(L)$ as shown in Figure 8. Any triangle in inside this rectangle has area at most half the area of the rectangle and therefore is at most $\overline{x}_{k_0}(L)\overline{y}_{k_0}(L)$. The only way that equality can hold is if the p_1 is the initial point of $\mathcal{C}$ and p_1 has coordinates $(-\overline{x}_{k_0}(L), \overline{y}_{k_0}(L))$, $p_2 = p_0$ is the midpoint of $\mathcal{C}$ and p_3 is the endpoint of $\mathcal{C}$ and has coordinates $(\overline{x}_{k_0}(L), \overline{y}_{k_0}(L))$. This implies equality holds in the upper bounds of Theorem 5.5 and therefore $\mathcal{C}$ has constant curvature k_0. $\square$

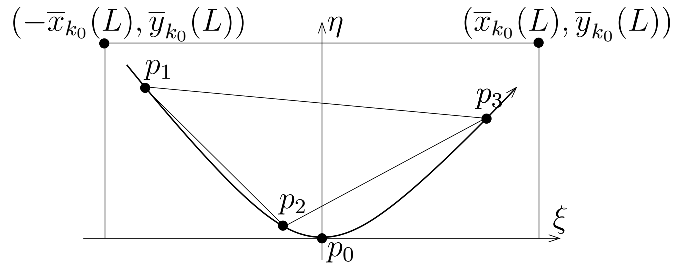

Figure 8. Let $L = \Lambda(\mathcal{C})/2$. Then Theorem 5.5 implies $\mathcal{C}$ is inside the pictured rectangle. Therefore $\triangle p_1p_2p_2$ has area at most half the area of this rectangle and equality only holds when p_1 is the upper left corner, p_3 is the upper right corner, and $p_1 = p_0$ is on the ξ-axis.

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References

[1] W. Blaschke. *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. II Affine Differentialgeometrie.* Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 1923.

[2] E. Calabi, P. J. Olver, and A. Tannenbaum. Affine geometry, curve flows, and invariant numerical approximations. *Adv. Math.*, 124(1):154–196, 1996.

[3] S. S. Chern. Curves and surfaces in Euclidean space. In *Studies in Global Geometry and Analysis*, pages 16–56. Math. Assoc. America, Buffalo, N.Y.; distributed by Prentice-Hall, Englewood Cliffs, N.J., 1967.

[4] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill Book Co., Inc., New York-Toronto-London, 1955.

[5] C. L. Epstein. The theorem of A. Schur in hyperbolic space. http://www.math.upenn.edu/~cle/papers/SchursLemma.pdf, 1985. Preprint, 46 pages.

[6] H. W. Guggenheimer. *Differential geometry*. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1977. Corrected reprint of the 1963 edition.

[7] G. B. Gustafson. Differential equations and linear algebra. http://www.math.utah.edu/~gustafso/debook/deBookGG.pdf, 1999—2022.

[8] D. L. Kreider, R. G. Kuller, D. R. Ostberg, and F. W. Perkins. *An introduction to linear analysis*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-Don Mills, Ont., 1966.

[9] R. López. The theorem of Schur in the Minkowski plane. *J. Geom. Phys.*, 61(1):342–346, 2011.

[10] A. Schur. Über die Schwarzsche Extremaleigenschaft des Kreises unter den Kurven konstanter Krümmung. *Math. Ann.*, 83(1-2):143–148, 1921.

[11] M. Spivak. *A comprehensive introduction to differential geometry. Vol. II*. Publish or Perish, Inc., Wilmington, Del., second edition, 1979.

[12] J. M. Sullivan. Curves of finite total curvature. In *Discrete differential geometry*, volume 38 of *Oberwolfach Semin.*, pages 137–161. Birkhäuser, Basel, 2008.

Department of Mathematics, University of South Carolina, Columbia, SC 29208

Email address: howard@math.sc.edu

URL: www.math.sc.edu/∼howard